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Excerpt
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Introduction: basics of QCD perturbation theory

Quantum chromodynamics (QCD) is the theory of strong interactions. This is an exciting physical theory, whose Lagrangian deals with quark and gluon fields and their interactions. At the same time, quarks and gluons do not exist as free particles in nature but combine into bound states (hadrons) instead. This phenomenon, known as *quark confinement*, is one of the most profound puzzles of QCD. Another amazing feature of QCD is the property of *asymptotic freedom*: quarks and gluons tend to interact more weakly over short distances and more strongly over longer distances.

This book is dedicated to another QCD mystery: the behavior of quarks and gluons in high energy collisions. Quantum chromodynamics is omnipresent in high energy collisions of all kinds of known particles. There are vast amounts of high energy scattering data on strong interactions, which have been collected at accelerators around the world. While these data are incredibly diverse they often exhibit intriguingly universal scaling properties, which unify much of the data while puzzling both experimentalists and theorists alike. Such universality appears to imply that the underlying QCD dynamics is the same for a broad range of high energy scattering phenomena.

The main goal of this book is to provide a consistent theoretical description of high energy QCD interactions. We will show that the QCD dynamics in high energy collisions is very sophisticated and often nonlinear. At the same time much solid theoretical progress has been made on the subject over the years. We will present the results of this progress by introducing a universal approach to a broad range of high energy scattering phenomena.

We begin by presenting a brief summary of the tools needed to perform perturbative QCD calculations. Since much of the material in this chapter is covered in standard field theory and particle physics textbooks, we will not derive many results, simply summarizing them and referring the reader to the appropriate literature for detailed derivations.

1.1 The QCD Lagrangian

Quantum chromodynamics is an SU(3) Yang–Mills gauge theory (Yang and Mills 1954) describing the interactions of quarks and gluons. The QCD Lagrangian density is

$$\mathcal{L}_{QCD} = \sum_{\text{flavors } f} \bar{q}_i^f(x) \left[i\gamma^\mu D_\mu - m_f \right]_{ij} q_j^f(x) - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \tag{1.1}$$

where $q_i^f(x)$ and $\bar{q}_i^f(x)$ are the quark and antiquark spin-1/2 Dirac fields of color i, flavor f, and mass m_f, with $\bar{q} = q^\dagger \gamma^0$. A field $A_\mu^a(x)$ describes the gluon, which has spin equal to 1, zero mass, and color index a in the adjoint representation of the SU(3) gauge group. Summation over repeated color and Lorentz indices is assumed, with $i, j = 1, 2, 3$ and $a = 1, \ldots, 8$. The covariant derivative D_μ is defined by

$$D_\mu = \partial_\mu - igA_\mu = \partial_\mu - igt^a A_\mu^a. \tag{1.2}$$

The t^a are the generators of SU(3) in the fundamental representation ($t^a = \lambda^a/2$, where the λ^a are the Gell-Mann matrices). The non-Abelian gluon field strength tensor $F_{\mu\nu}^a$ is defined by

$$F_{\mu\nu} = t^a F_{\mu\nu}^a = \frac{i}{g}\left[D_\mu, D_\nu\right] \tag{1.3}$$

or, equivalently, by

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c, \tag{1.4}$$

where f^{abc} are the structure constants of the color group SU(3).

We work in natural units, with $\hbar = c = 1$. Our four-vectors are $x^\mu = (t, \vec{x})$, the partial derivatives are denoted $\partial_\mu = \partial/\partial x^\mu$, and the metric in t, x, y, z coordinates is $g_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$.

The Lagrangian of Eq. (1.1) was proposed by Fritzsch, Gell-Mann, and Leutwyler (1973), Gross and Wilczek (1973, 1974), and Weinberg (1973). The form of the QCD Lagrangian is based on two assumptions confirmed by experimental observations: all hadrons consist of quarks and quarks cannot be observed as free particles. The first observation leads to a new quantum number for quarks: color. Indeed, without this quantum number we cannot build the wave functions for baryons. For example the Ω^- hyperon has spin 3/2 and consists of three s-quarks. This means that the spin and flavor parts of its wave function are symmetric with respect to interchange of the identical valence s-quarks. Owing to the Pauli exclusion principle the full wave function of the three identical quarks has to be antisymmetric. If spin and flavor were the only quantum numbers, it would appear that the spatial wave function of the three s-quarks would have to be antisymmetric. However, this would contradict the fact that Ω^- is a stable particle and is, therefore, a ground state of the three s-quark system. The spatial wave function of a ground state has to be symmetric. To resolve this conundrum we need to introduce a new quantum number that should have at least three different values to make the three strange quarks different in the Ω^- hyperon. This quantum number is the quark *color*.

We then need to determine which particle is responsible for interactions between the quarks forming quark bound states, the hadrons. The interactions between the quarks in mesons and baryons have to be attractive, which indicates that they should depend on quark color: if one introduced interactions between quarks using some global (not gauged) non-Abelian color symmetry then one would not be able to obtain attractive interactions between the quark and the antiquark in a meson and between a pair of quarks in a baryon simultaneously, at least not in the lowest nontrivial order in the interaction. One therefore concludes that the non-Abelian color symmetry has to be gauged by introducing a non-Abelian vector boson responsible for quark interactions. Moreover, as we will see below, the high energy scattering data confirms this conclusion as it demonstrates that the particle responsible for quark interactions has spin equal to 1.

The second experimental observation needed for the construction of the QCD Lagrangian, that quarks are never seen as free particles, means that the forces between quarks should be stronger at longer distances to prevent quarks from leaving a hadron. For point-like particles our best chance of getting such forces is by assuming that quark interactions are mediated by a massless particle. For such a particle the lowest-order quark–antiquark interaction potential decreases at long distances roughly as to $1/r$, where r is the distance between the quarks. (Indeed in a full QCD calculation this behavior changes to $\sim r$, that of a confining potential.) Massive particles would give an exponentially decreasing potential, which would have a shorter range than the potential in the massless case. We therefore conclude that the particle responsible for quark interactions is a non-Abelian massless vector boson, a gluon.

However, particle interactions may generate a mass even for a particle that is massless at the Lagrangian level. To protect the zero mass of the gluon from higher-order corrections we have to assume the existence of gauge symmetry in our Lagrangian. Namely, the Lagrangian should be invariant with respect to

$$q(x) \to S(x)\, q(x), \tag{1.5a}$$

$$\bar{q}(x) \to \bar{q}(x)\, S^{-1}(x), \tag{1.5b}$$

$$A_\mu(x) \to S(x) A_\mu(x) S^{-1}(x) - \frac{i}{g}\left[\partial_\mu S(x)\right] S^{-1}(x), \tag{1.5c}$$

where we have defined a unitary 3×3 matrix

$$S(x) = e^{i\alpha^a(x)\, t^a}, \tag{1.6}$$

where the $\alpha^a(x)$ are arbitrary real-valued functions; summation over repeated color indices a is again implied. The form of the Yang–Mills Lagrangian (1.1) can be derived directly from the gauge symmetry in Eqs. (1.5) (see e.g. Peskin and Schroeder (1995)).

1.2 A review of Feynman rules for QCD

To derive the Feynman rules from the Lagrangian (1.1) we need to define the functional integral (the QCD partition function)

$$Z_{QCD} = \int \mathcal{D}A\, \mathcal{D}q\, \mathcal{D}\bar{q}\, \exp\left\{ i \int d^4x\, \mathcal{L}_{QCD}(A, q, \bar{q}) \right\}. \tag{1.7}$$

One can see that this integral is divergent since its integrand has the same value for an infinite set of fields related to each other by all possible gauge transformations (1.5). However, the values of physical observables are given by the expectation values of operators. For an arbitrary gauge-invariant operator $\mathcal{O}$ we have the vacuum expectation value

$$\langle \mathcal{O} \rangle \equiv \frac{\int \mathcal{D}A\, \mathcal{D}q\, \mathcal{D}\bar{q}\, \mathcal{O} \exp\left\{i \int d^4x \mathcal{L}_{QCD}\right\}}{\int \mathcal{D}A\, \mathcal{D}q\, \mathcal{D}\bar{q} \exp\left\{i \int d^4x\, \mathcal{L}_{QCD}\right\}} \tag{1.8}$$

The divergences caused by integrations over gauge directions in the numerator and in the denominator of Eq. (1.8) cancel each other. Faddeev and Popov (1967) suggested a procedure allowing one to see such cancellations in the most economic way by multiplying the definition (1.7) with the functional integral identity[1]

$$1 = \int \mathcal{D}\alpha\, \delta(\alpha) = \int \mathcal{D}\alpha\, \delta(G(A^\alpha))\, \det\left(\frac{\delta G(A^\alpha)}{\delta\, \alpha}\right), \tag{1.9}$$

where the integral runs over all gauge transformations labeled by α^a (see Eq. (1.6)), A^α is a gauge field related to the original one by the gauge transformation defined by α^a, and $G(A) = 0$ is the gauge-fixing condition. (For instance, $G(A) = \partial_\mu A^\mu$ in a covariant gauge.) Let us restrict ourselves to gauges in which the functional determinant $\det[\delta G(A^\alpha)/\delta\alpha]$ is independent of α^a for a given A^α. Using Eq. (1.9) the expectation values of the operators can be written as

$$\langle \mathcal{O} \rangle = \frac{\left(\int \mathcal{D}\alpha\right) \int \mathcal{D}A\, \mathcal{D}q\, \mathcal{D}\bar{q}\, \mathcal{O}\; \delta(G(A))\, \det\left(\frac{\delta G(A^\alpha)}{\delta\,\alpha}\right) \exp\left\{i \int d^4x\, \mathcal{L}_{QCD}\right\}}{\left(\int \mathcal{D}\alpha\right) \int \mathcal{D}A\, \mathcal{D}q\, \mathcal{D}\bar{q}\, \delta\left(G(A)\right)\, \det\left(\frac{\delta G(A^\alpha)}{\delta\,\alpha}\right) \exp\left\{i \int d^4x\, \mathcal{L}_{QCD}\right\}}, \tag{1.10}$$

where we have relabeled the integration variable A^α as A everywhere except in the determinants, in which one should put $\alpha^a = 0$ after differentiation thus turning A^α into A. The infinities in the numerator and the denominator of Eq. (1.10) are clearly identifiable as being due to the integration over α^a. As nothing else in the integrands of Eq. (1.10) depends on α we can simply cancel the $\mathcal{D}\alpha$ integrations, writing

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}A\, \mathcal{D}q\, \mathcal{D}\bar{q}\, \mathcal{O}\; \delta(G(A))\, \det\left(\frac{\delta G(A^\alpha)}{\delta\,\alpha}\right) \exp\left\{i \int d^4x\, \mathcal{L}_{QCD}\right\}}{\int \mathcal{D}A\, \mathcal{D}q\, \mathcal{D}\bar{q}\, \delta(G(A))\, \det\left(\frac{\delta G(A^\alpha)}{\delta\,\alpha}\right) \exp\left\{i \int d^4x\, \mathcal{L}_{QCD}\right\}}. \tag{1.11}$$

To obtain the Feynman rules we have to put all the A-dependence in the integrands in Eq. (1.11) into the exponents. We start with the delta functions and first note that making the replacement in Eq. (1.11)

$$\delta(G(A)) \to \delta(G(A) - r(x)), \tag{1.12}$$

where $r(x)$ is some arbitrary function of x^μ, would not change the values of the functional integrals in the numerator and the denominator and would therefore leave $\langle \mathcal{O} \rangle$ unchanged. Indeed different choices of $r(x)$ correspond to different choices of the gauge defined by the $G(A) = r(x)$ gauge condition. Thus the replacement (1.12) simply modifies the function defining the gauge condition: $G(A) \to G(A) - r(x)$. Since our initial gauge-defining function $G(A)$ is arbitrary, and as neither of the integrals in the numerator and the denominator of Eq. (1.11) depends on $G(A)$, we conclude that nothing in the numerator

[1] In discussing the Faddeev–Popov method we will follow closely the presentations in Peskin and Schroeder (1995) and in Sterman (1993).

or the denominator of Eq. (1.11) changes if we perform the replacement (1.12). Moreover, the resulting expression,

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}A\, \mathcal{D}q\, \mathcal{D}\bar{q}\, \mathcal{O}\, \delta(G(A) - r(x))\ \det\left(\frac{\delta G(A^\alpha)}{\delta \alpha}\right)\ \exp\left\{i \int d^4x\, \mathcal{L}_{QCD}\right\}}{\int \mathcal{D}A\, \mathcal{D}q\, \mathcal{D}\bar{q}\, \delta(G(A) - r(x))\ \det\left(\frac{\delta G(A^\alpha)}{\delta \alpha}\right)\ \exp\left\{i \int d^4x\, \mathcal{L}_{QCD}\right\}}, \tag{1.13}$$

is independent of $r(x)$ for the same reasons. We can integrate the numerator and the denominator separately over $r(x)$ by multiplying them with

$$1 = N(\xi) \int \mathcal{D}r\ \exp\left\{-i \int d^4x\, \frac{r^2(x)}{2\xi}\right\}, \tag{1.14}$$

where $N(\xi)$ is a normalization function defined by Eq. (1.14) and ξ is an arbitrary number. Multiplying both the numerator and the denominator of Eq. (1.13) by Eq. (1.14), canceling $N(\xi)$, and performing the r-integrals with the help of the delta functions, we obtain

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}A\, \mathcal{D}q\, \mathcal{D}\bar{q}\, \mathcal{O} \det\left(\frac{\delta G(A^\alpha)}{\delta \alpha}\right)\ \exp\left\{i \int d^4x\, \left(\mathcal{L}_{QCD} - \frac{1}{2\xi}\, [G(a)]^2\right)\right\}}{\int \mathcal{D}A\, \mathcal{D}q\, \mathcal{D}\bar{q} \det\left(\frac{\delta G(A^\alpha)}{\delta \alpha}\right)\ \exp\left\{i \int d^4x\, \left(\mathcal{L}_{QCD} - \frac{1}{2\xi}\, [G(a)]^2\right)\right\}}. \tag{1.15}$$

Finally, in order to remove the determinants of Eq. (1.15) into the exponents one introduces the (unphysical) *Faddeev–Popov ghost field* $c^a(x)$, whose values are complex Grassmann numbers (Faddeev and Popov 1967, Feynman 1963, DeWitt 1967). The ghost field is a Lorentz scalar in the adjoint representation of SU(3). With the help of the Faddeev–Popov ghost field we write

$$\det\left(\frac{\delta G\,(A^\alpha)}{\delta\, \alpha}\right) = \int \mathcal{D}c\, \mathcal{D}c^*\ \exp\left\{-i \int d^4x\, c^*\, \frac{\delta G\,(A^\alpha)}{\delta\, \alpha}\, c\right\} \tag{1.16}$$

with c^* the complex conjugate of the c field. Using Eq. (1.16) in Eq. (1.15) we obtain

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{D}A\, \mathcal{D}q\, \mathcal{D}\bar{q}\, \mathcal{D}c\, \mathcal{D}c^*\, \mathcal{O}\ \exp\left\{i \int d^4x\, \mathcal{L}(A, q, \bar{q}, c, c^*)\right\}}{\int \mathcal{D}A\, \mathcal{D}q\, \mathcal{D}\bar{q}\, \mathcal{D}c\, \mathcal{D}c^*\ \exp\left\{i \int d^4x\, \mathcal{L}(A, q, \bar{q}, c, c^*)\right\}}, \tag{1.17}$$

where we have defined an effective Lagrangian

$$\mathcal{L}(A, q, \bar{q}, c, c^*) \equiv \mathcal{L}_{QCD} - \frac{1}{2\xi}\, [G(A)]^2 - c^*\, \frac{\delta G(A^\alpha)}{\delta\, \alpha}\, c. \tag{1.18}$$

Now we are ready to derive the Feynman rules for QCD.

In this book we will employ two main gauge choices. One is the Lorenz gauge, defined by the gauge condition

$$\partial_\mu A^{a\,\mu} = 0. \tag{1.19}$$

Inserting $G(A) = \partial_\mu A^{a\,\mu}$ into Eq. (1.18), after some straightforward algebra (see e.g. Peskin and Schroeder (1995)) we end up with

$$\mathcal{L} = \mathcal{L}_{QCD} - \frac{1}{2\xi}\, \left(\partial^\mu A^a_\mu\right)^2 + \left(\partial^\mu c^{a\,*}\right)\left(\delta^{ac}\, \partial^\mu + g f^{abc} A^b_\mu\right) c^c. \tag{1.20}$$

Using Eq. (1.20) we can derive the Feynman rules for QCD by substituting the Lagrangian (1.20) into Eq. (1.7) in place of $\mathcal{L}_{QCD}$.

The other gauge choice that we will be using frequently throughout the book is the light cone gauge, defined by

$$\eta \cdot A^a = \eta^\mu A^a_\mu = 0, \tag{1.21}$$

with η^μ a constant four-vector that is light-like, so that $\eta^2 = \eta_\mu\,\eta^\mu = 0$. One can show that, in the light cone gauge, $\det[\delta G\,(A^\alpha)/\delta\,\alpha]$ does not depend on A^μ when we take the limit $\xi \to 0$. From Eq. (1.18) one can see that in this case the ghost field would not couple to the gluon field and so can be integrated out in the functional integrals of Eq. (1.17). Hence there is no ghost field in the light cone gauge. The effective Lagrangian (1.18) in the light cone gauge becomes

$$\mathcal{L} = \mathcal{L}_{QCD} - \frac{1}{2\,\xi}\left(\eta^\mu A^a_\mu\right)^2 \tag{1.22}$$

(with an implied $\xi \to 0$ limit).

Below we list the Feynman rules for QCD, in the Lorenz and light cone gauges, which follow from the Lagrangians in Eqs. (1.20) and (1.22). We use the standard notation for a product of two four-vectors $u \cdot v = u_\mu v^\mu$ and for the square of a single four-vector $v_\mu v^\mu = v^2$. The Dirac gamma matrices in the standard Dirac representation, which we will use here, are defined by

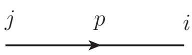
$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{1.23}$$

where $\mathbf{1}$ is a unit 2×2 matrix, $i = 1, 2, 3$, and σ^i are the Pauli matrices

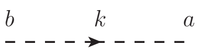
$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.24}$$

As usual, we will write $\not{v} = \gamma^\mu v_\mu$. Arrows on the quark and ghost propagators (see below) indicate the flow of the particle number and, in the cases of the quark propagator and the ghost–gluon vertex, they also indicate the momentum flow. As ghost fields do not exist in the light cone gauge, the Feynman rules for ghosts listed below apply only in the Lorenz gauge.

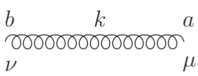
1.2.1 QCD Feynman rules

Quark propagator: 

$$= \frac{i(\not{p} + m_f)}{p^2 - m_f^2 + i\epsilon}\,\delta^{ij}, \tag{1.25}$$

Ghost propagator: 

$$= \frac{i}{k^2 + i\epsilon}\,\delta^{ab}, \tag{1.26}$$

Gluon propagator: 

$$= \frac{-i\,D_{\mu\nu}(k)}{k^2 + i\epsilon}\,\delta^{ab}, \tag{1.27}$$

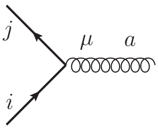
where in the Lorenz gauge ($\partial \cdot A^a = 0$)

$$D_{\mu\nu}(k) = g_{\mu\nu} - (1 - \xi)\,\frac{k_\mu k_\nu}{k^2}; \tag{1.28}$$

the choice $\xi = 0$ is referred to as the Landau gauge and the choice $\xi = 1$ is called the Feynman gauge. In the light cone gauge $\eta \cdot A^a = 0$ with $\xi \to 0$ one has

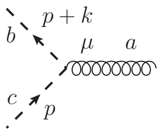
$$D_{\mu\nu}(k) = g_{\mu\nu} - \frac{\eta_\mu k_\nu + \eta_\nu k_\mu}{\eta \cdot k}. \tag{1.29}$$

Quark–gluon vertex:



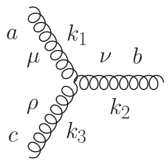
$$= i g \gamma^\mu \,(t^a)_{ji}, \tag{1.30}$$

Ghost–gluon vertex (Lorenz gauge only):



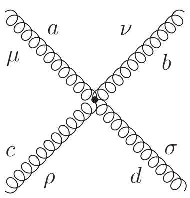
$$= g(p+k)^\mu f^{abc} \tag{1.31}$$

Three-gluon vertex (all momenta flow into the vertex):



$$= -g f^{abc}\,[(k_1 - k_3)^\nu g^{\mu\rho} + (k_2 - k_1)^\rho g^{\mu\nu} + (k_3 - k_2)^\mu g^{\nu\rho}] \tag{1.32}$$

Four-gluon vertex:



$$= -i g^2 \left[f^{abe}\, f^{cde}\,(g^{\mu\rho}\, g^{\nu\sigma} - g^{\mu\sigma}\, g^{\nu\rho}) + f^{ace}\, f^{bde}\,(g^{\mu\nu}\, g^{\rho\sigma} - g^{\mu\sigma}\, g^{\nu\rho}) + f^{ade}\, f^{bce}\,(g^{\mu\nu}\, g^{\rho\sigma} - g^{\mu\rho}\, g^{\nu\sigma}) \right] \tag{1.33}$$

The Feynman rules that are standard for all field theories, such as the conservation of four-momentum in the vertices and the inclusion of a factor -1 for each fermion loop or of proper symmetry factors, apply to QCD as well and will not be explicitly spelled out here.

1.3 Rules of light cone perturbation theory

Many calculations in this book will not be performed using the Feynman rules. Instead we will use light cone perturbation theory (LCPT), following the rules introduced by Lepage and Brodsky (1980) (see Brodsky and Lepage (1989) and Brodsky, Pauli, and Pinsky (1998) for a detailed derivation of the LCPT rules). We begin by introducing the light cone notation.

For any four-vector v^μ we define

$$v^+ = v^0 + v^3, \quad v^- = v^0 - v^3. \tag{1.34}$$

With this notation we see immediately that

$$v^2 = v^+ v^- - \vec{v}_\perp^{\,2}, \tag{1.35}$$

where we have defined a vector of transverse components $\vec{v}_\perp = (v^1, v^2)$. A product of two four-vectors v^μ and u^μ in light cone notation is

$$u \cdot v = \frac{1}{2} u^+ v^- + \frac{1}{2} u^- v^+ - \vec{u}_\perp \cdot \vec{v}_\perp. \tag{1.36}$$

The metric has nonzero components $g_{+-} = g_{-+} = 1/2$, $g_{11} = g_{22} = -1$. This gives

$$v_- = \frac{v^0 + v^3}{2} = \frac{v^+}{2}, \quad v_+ = \frac{v^0 - v^3}{2} = \frac{v^-}{2}. \tag{1.37}$$

Note also that $\partial_+ = (1/2)\,\partial^-$ and $\partial_- = (1/2)\,\partial^+$.

Light cone perturbation theory is similar to time-ordered perturbation theory, except that the light cone x^+-direction plays the role of time. (For a good presentation of time-ordered perturbation theory see Sterman (1993).) Our discussion of LCPT here will closely follow Lepage and Brodsky (1980) and Brodsky and Lepage (1989). We will work in the particular light cone gauge

$$A^+ = 0, \tag{1.38}$$

which can be obtained from Eq. (1.21) by choosing $\eta^\mu = (0, 2, \vec{0}_\perp)$, in the $(+, -, \perp)$ notation. Of the remaining A^- and $A^i_\perp$ components of the gluon field ($i = 1, 2$), only the transverse components $A^i_\perp$ are independent. The component A^- can be expressed in terms of the $A^i_\perp$ using the equations of motion for the QCD Lagrangian (1.1). The quark field, which we will denote by $q(x)$, dropping the flavor label, is separated into two spinor components q_+ and q_- defined by

$$q_\pm(x) = \Lambda_\pm\, q(x), \tag{1.39}$$

where the projection operators $\Lambda_\pm$ are given by

$$\Lambda_\pm = \frac{1}{2} \gamma^0 \gamma^\pm \tag{1.40}$$

and the Dirac matrix $\gamma^\pm = \gamma^0 \pm \gamma^3$. Note that, just like any other projection operators, $\Lambda_\pm$ obey the following relations: $\Lambda_+ \Lambda_- = 0$, $\Lambda_\pm^2 = \Lambda_\pm$, and $\Lambda_+ + \Lambda_- = 1$. The two projections q_+ and q_- are not independent and can also be related using the constraint part of the equations of motion. The dependent field operators A^- and q_- are expressed in terms of $A_\perp^i$ and q_+ as (see Lepage and Brodsky (1980))[2]

$$A^- = -\frac{2}{\partial^+} \partial_{\perp j} \cdot A_\perp^j + \frac{2g}{(\partial^+)^2} \left\{ \left[i\partial^+ A_\perp^j, A_\perp^j \right] + 2 q_+^\dagger t^a q_+ t^a \right\}, \tag{1.41}$$

$$q_- = \frac{1}{i\partial^+} \gamma^0 \left(-i\gamma_\perp^j D_{\perp j} + m \right) q_+ \tag{1.42}$$

where $j = 1, 2$. Next one defines free gluon and quark fields $\tilde{A}^\mu$ and $\tilde{q}$ by

$$\tilde{A}^\mu = (0, \tilde{A}^-, \vec{A}_\perp), \tag{1.43}$$

in the $(+, -, \perp)$ notation, with

$$\tilde{A}^- \equiv -\frac{2}{\partial^+} \partial_{\perp j} \cdot A_\perp^j \tag{1.44}$$

and

$$\tilde{q} \equiv q_+ + \frac{1}{i\partial^+} \gamma^0 \left(-i\gamma_\perp^j \partial_{\perp j} + m \right) q_+. \tag{1.45}$$

The light cone Hamiltonian H is defined as the minus component of the four-momentum vector, P^-. It can be written as the sum of free and interaction terms:

$$H = P^- = H_0 + H_{int}, \tag{1.46}$$

where (Lepage and Brodsky 1980, Brodsky and Lepage 1989, Brodsky, Pauli, and Pinsky 1998)

$$H_0 = \frac{1}{2} \int dx^- d^2x_\perp \left(\bar{\tilde{q}} \gamma^+ \frac{m^2 - \nabla_\perp^2}{i\partial^+} \tilde{q} - \tilde{A}_\mu^a \nabla_\perp^2 \tilde{A}^{a\,\mu} \right) \tag{1.47}$$

is the free part of the Hamiltonian, while the interaction part is given by

$$\begin{aligned} H_{int} = \int dx^- d^2x_\perp \Big[& -2g \,\mathrm{tr}\left(i\partial^\mu \tilde{A}^\nu [\tilde{A}_\mu, \tilde{A}_\nu] \right) - \frac{g^2}{2} \mathrm{tr}\left([\tilde{A}^\mu, \tilde{A}^\nu][\tilde{A}_\mu, \tilde{A}_\nu] \right) \\ & - g\bar{\tilde{q}}\gamma^\mu A_\mu \tilde{q} + g^2 \,\mathrm{tr}\left([i\partial^+ \tilde{A}^\mu, \tilde{A}_\mu] \frac{1}{(i\partial^+)^2} [i\partial^+ \tilde{A}^\nu, \tilde{A}_\nu] \right) \\ & + g^2 \bar{\tilde{q}} \gamma^\mu A_\mu \gamma^+ \frac{1}{2i\partial^+} \gamma^\nu A_\nu \tilde{q} - g^2 \bar{\tilde{q}} \gamma^+ \left(\frac{1}{(i\partial^+)^2} [i\partial^+ \tilde{A}^\mu, \tilde{A}_\mu] \right) \tilde{q} \\ & + \frac{g^2}{2} \bar{\tilde{q}} \gamma^+ t^a q \frac{1}{(i\partial^+)^2} \bar{\tilde{q}} \gamma^+ t^a q \Big]. \end{aligned} \tag{1.48}$$

Quantizing the theory by expanding $A_\perp^i$ and q_+ in terms of creation and annihilation operators while treating the x^+ light cone direction as time, one can construct light cone time-ordered perturbation theory with the help of the light cone Hamiltonian H. The rules of LCPT for the calculation of scattering amplitudes are given in the following subsection (Lepage and Brodsky 1980, Brodsky and Lepage 1989, Zhang and Harindranath 1993, Brodsky, Pauli, and Pinsky 1998).

[2] Our notation in Eqs. (1.1), (1.2), and (1.4), and therefore throughout the book, can be obtained from that of Lepage and Brodsky (1980) and Brodsky and Lepage (1989) by making the replacement $g \to -g$.

1.3.1 QCD LCPT rules

1. Draw all diagrams for a given process at the desired order in the coupling constant, including all possible orderings of the interaction vertices in the light cone time x^+. Assign a four-momentum k^μ to each line such that it is on mass shell, so that $k^2 = m^2$ with m the mass of the particle. Each vertex conserves only the k^+ and $\vec{k}_\perp$ components of the four-momentum. Hence for each line the four-momentum has components as follows:

$$k^\mu = \left(k^+, \frac{\vec{k}_\perp^2 + m^2}{k^+}, \vec{k}_\perp^2 \right). \tag{1.49}$$

2. With quarks associate on-mass-shell spinors in the Lepage and Brodsky (1980) convention:

$$u_\sigma(p) = \frac{1}{\sqrt{p^+}} \left(p^+ + m\gamma^0 + \gamma^0 \vec{\gamma}_\perp \cdot \vec{p}_\perp \right) \chi(\sigma), \tag{1.50}$$

$$v_\sigma(p) = \frac{1}{\sqrt{p^+}} \left(p^+ - m\gamma^0 + \gamma^0 \vec{\gamma}_\perp \cdot \vec{p}_\perp \right) \chi(-\sigma), \tag{1.51}$$

with

$$\chi(+1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \chi(-1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}. \tag{1.52}$$

Gluon lines come with a polarization vector $\epsilon_\lambda^\mu(k)$. In the $A^+ = 0$ gauge this vector is given by

$$\epsilon_\lambda^\mu(k) = \left(0, \frac{2\,\vec{\epsilon}_\perp^{\,\lambda} \cdot \vec{k}_\perp}{k^+}, \vec{\epsilon}_\perp^{\,\lambda} \right) \tag{1.53}$$

with transverse polarization vector

$$\vec{\epsilon}_\perp^{\,\lambda} = -\frac{1}{\sqrt{2}} (\lambda, i), \tag{1.54}$$

where $\lambda = \pm 1$. Equation (1.53) follows from requiring that $\epsilon_\lambda^+ = 0$ and $\epsilon_\lambda(k) \cdot k = 0$.

3. For each intermediate state there is a factor equal to the light cone energy denominator

$$\frac{1}{\sum\limits_{inc} k^- - \sum\limits_{interm} k^- + i\,\epsilon} \tag{1.55}$$

where the sums run respectively over all incoming particles present in the initial state in the diagram ("inc") and over all the particles in the intermediate state at hand ("interm"). According to rule 1 above, for each particle we have $k^- = (\vec{k}_\perp^2 + m^2)/k^+$. Since the k^- momentum component is not conserved at the vertices the intermediate states are not on the "energy shell" and the light cone denominator in (1.55) is nonzero. Note that the light