Characters of Crossed Modules and
Premodular Categories

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Abstract

A general procedure is presented, associating a premodular category to a finite crossed module, generalizing the representation category of the double of a finite group, and the extent to which the resulting premodular category fails to be modular is explained.

1. Introduction

Modular Tensor Categories (MTCs for short) [1, 16] have attracted much attention in recent years, which is due to the recognition of their importance in both pure mathematics – 3-dimensional topology, representations of Vertex Operator Algebras (VOAs for short) – and theoretical physics (Rational Conformal Field Theory, Topological Field Theories). They are also closely related to Moonshine [4, 7, 10]: a most interesting (and mysterious) example of a Modular Tensor Category, which is responsible for some of the deeper aspects of Moonshine, is the MTC associated to the Moonshine orbifold, i.e. the fixed point VOA of the Moonshine module under the action of the Monster: note that this MTC is yet to be rigorously constructed.

As in every branch of science, a deeper understanding of Modular Tensor Categories requires a suitable supply of examples. Since the work of Huang [12], we know that the module category of any rational VOA (satisfying some technical conditions) is modular, but this important result doesn't help us that much, because VOAs are pretty complicated objects usually hard to deal with. This leads to the desire of associating MTCs to simpler and more accessible algebraic objects. There are several such constructions, a most notable case being the one that associates to a finite group the module category of its

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(Drinfeld) double [2, 8]. The aim of the present note is to sketch a generalization of this last construction, associating to any (finite) crossed module a premodular category, i.e. a braided tensor category that falls short of being modular. The idea behind is to use 'higher dimensional groups', whose simplest instance are crossed modules [5, 18], for constructing Modular Tensor Categories. In the sequel we will examine to which extent this idea may be put to work.

The plan of the paper is the following. In the next section we'll recall some basic definitions and results about crossed modules. In Section 3 we introduce our basic object of study, the tensor category associated to the crossed module, and discuss some of its properties. Section 4 describes the notion of characters of crossed modules, the main technical tool in our study. Section 5 discusses the premodular structure of the category, and the extent to which it fails to be modular. We conclude by some remarks on the possible applications of the results presented.

We have decided to present only an outline of the theory, without going into detailed proofs, since we felt that their inclusion would not help to clarify the arguments, but could hide the main line of thought. Detailed proofs of all the results to be presented could be supplied by exploiting the close analogy with the character theory of finite groups.

2. Crossed modules

To begin with, let's recall that an action of the group G on the group M is a homomorphism $G \to \mathrm{Aut}\,(M)$ or, what is the same, a map $\mu : M \times G \to M$ such that

(1) $\mu\,(m_1 m_2, g) = \mu\,(m_1, g)\,\mu\,(m_2, g)$ for all $m_1, m_2 \in M$ and $g \in G$;
(2) $\mu\,(m, g_1 g_2) = \mu\,(\mu\,(m, g_1)\,, g_2)$ for all $m \in M$ and $g_1, g_2 \in G$.

As is customary, we'll use the exponential notation $\mu\,(m, g) = m^g$ in the sequel.

A crossed module [5, 11, 18] is nothing but a 4-tuple $\mathcal{X} = (\mathcal{X}_1, \mathcal{X}_2, \mu, \partial)$, where $\mathcal{X}_1, \mathcal{X}_2$ are groups, μ is an action of $\mathcal{X}_1$ on $\mathcal{X}_2$, and $\partial : \mathcal{X}_2 \to \mathcal{X}_1$ is a homomorphism, called the boundary map, that satisfies

XMod1: $\partial\,(m^g) = g^{-1}\,(\partial m)\,g$ for all $m \in \mathcal{X}_2$ and $g \in \mathcal{X}_1$;
XMod2: $m^{\partial n} = n^{-1} m n$ for all $m, n \in \mathcal{X}_2$.

A crossed module is finite if both $\mathcal{X}_1$ and $\mathcal{X}_2$ are finite groups. Examples of crossed modules abound in algebra and topology, let's just cite two, coming from group theory, that will guide our investigations later.

Example 1. For a group G, we'll denote by $\mathcal{R}G$ the crossed module $(G, \mathbf{1}, \mu, \partial)$, where $\mathbf{1}$ denotes the trivial subgroup of G, i.e. $\mathbf{1} = \{1\}$, and both the action μ and the boundary map ∂ are trivial.

Example 2. If G is a group, $\mathcal{D}G$ is the crossed module $(G, G, \mu, \mathbf{id})$, where μ is the conjugation action, i.e. $\mu(m, g) = g^{-1}mg$, and $\mathbf{id} : g \mapsto g$ is the trivial map.

A standard consequence of the defining properties of a crossed module is that $K = \ker \partial$ is a central subgroup of $\mathcal{X}_2$, $I = \mathrm{im}\, \partial$ is a normal subgroup of $\mathcal{X}_1$, and one has an exact sequence

$$1 \to K \to \mathcal{X}_2 \to \mathcal{X}_1 \to C \to 1 \tag{1}$$

where $C = \mathcal{X}_1/I$ is the cokernel of ∂ [5]. In particular, $|\mathcal{X}_2||C| = |K||\mathcal{X}_1|$ for a finite crossed module.

Finally, a morphism $\phi : \mathcal{X} \to \mathcal{Y}$ between the crossed modules $\mathcal{X} = (\mathcal{X}_1, \mathcal{X}_2, \mu_\mathcal{X}, \partial_\mathcal{X})$ and $\mathcal{Y} = (\mathcal{Y}_1, \mathcal{Y}_2, \mu_\mathcal{Y}, \partial_\mathcal{Y})$ is a pair (ϕ_1, ϕ_2), where $\phi_i : \mathcal{X}_i \to \mathcal{Y}_i$ are group homomorphisms for $i = 1, 2$, and the following relations hold:

$$\partial_\mathcal{Y} \circ \phi_2 = \phi_1 \circ \partial_\mathcal{X}$$
$$\mu_\mathcal{Y} \circ (\phi_2 \times \phi_1) = \phi_2 \circ \mu_\mathcal{X} ,$$

which simply express the commutativity of the diagrams

$$
\begin{array}{ccc}
\mathcal{X}_2 & \xrightarrow{\partial_\mathcal{X}} & \mathcal{X}_1 \\
\phi_2 \downarrow & & \downarrow \phi_1 \\
\mathcal{Y}_2 & \xrightarrow{\partial_\mathcal{Y}} & \mathcal{Y}_1
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathcal{X}_2 \times \mathcal{X}_1 & \xrightarrow{\mu_\mathcal{X}} & \mathcal{X}_2 \\
\phi_2 \times \phi_1 \downarrow & & \downarrow \phi_2 \\
\mathcal{Y}_2 \times \mathcal{Y}_1 & \xrightarrow{\mu_\mathcal{Y}} & \mathcal{Y}_2
\end{array}
$$

3. The category

To any finite crossed module $\mathcal{X} = (\mathcal{X}_1, \mathcal{X}_2, \mu, \partial)$ we'll associate a braided tensor category $\mathcal{M}(\mathcal{X})$, which falls short of being modular. Let's begin by describing the objects and morphisms of $\mathcal{M}(\mathcal{X})$. Here and in the sequel, we use the notation

$$\delta(x, y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

An object of $\mathcal{M}(\mathcal{X})$ is a triple (V, P, Q), where V is a complex linear space, while P and Q are maps $P : \mathcal{X}_2 \to \mathrm{End}(V)$ and $Q : \mathcal{X}_1 \to \mathrm{GL}(V)$ such that for all $g, h \in \mathcal{X}_1$ and $m, n \in \mathcal{X}_2$

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$$P(m) P(n) = \delta(m, n) P(m) \tag{2}$$

$$\sum_{m \in \mathcal{X}_2} P(m) = \mathbf{id}_V \tag{3}$$

$$Q(g) Q(h) = Q(gh) \tag{4}$$

$$P(m) Q(g) = Q(g) P(m^g) \tag{5}$$

By the dimension of an object (V, P, Q) we'll mean the dimension of the linear space V. A morphism $\phi : (V_1, P_1, Q_1) \to (V_2, P_2, Q_2)$ between two objects of $\mathcal{M}(\mathcal{X})$ is a linear map $\phi : V_1 \to V_2$ such that $\phi \circ P_1(m) = P_2(m) \circ \phi$ for all $m \in \mathcal{X}_2$ and, $\phi \circ Q_1(g) = Q_2(g) \circ \phi$ for all $g \in \mathcal{X}_1$. In general, we won't distinguish isomorphic objects of $\mathcal{M}(\mathcal{X})$.

Let's look at a couple of illustrating examples of objects of $\mathcal{M}(\mathcal{X})$ for a finite crossed module $\mathcal{X} = (\mathcal{X}_1, \mathcal{X}_2, \mu, \partial)$.

Example 3. The triple $\underline{\mathbf{1}} = (V, P, Q)$, with $V = \mathbb{C}$, $P(m) = \delta(m, 1) \mathbf{id}_V$ and $Q(g) = \mathbf{id}_V$, is a one dimensional object of $\mathcal{M}(\mathcal{X})$, that we'll call the trivial object.

Example 4. The triple $\underline{\mathbf{R}} = (V, P, Q)$, with $V = \mathbb{C}(\mathcal{X}_1 \times \mathcal{X}_2)$ and $P(m) \phi :$ $(x, y) \mapsto \delta(m, y^x) \phi(x, y)$, $Q(g) \phi : (x, y) \mapsto \phi(xg, y)$ for $\phi \in V$ and $(x, y) \in \mathcal{X}_1 \times \mathcal{X}_2$, is an object of $\mathcal{M}(\mathcal{X})$, that we'll call the regular object. Clearly, $\dim \underline{\mathbf{R}} = |\mathcal{X}_1| |\mathcal{X}_2|$.

Example 5. The triple $\underline{\mathbf{0}} = (V, P, Q)$, with $V = \mathbb{C}(K \times C)$ (remember the notations $K = \ker \partial$, $I = \operatorname{im} \partial$ and $C = \operatorname{coker} \partial = \mathcal{X}_1/I$ from Eq.1) and $P(m) \phi : (x, Iy) \mapsto \delta(m, x^y) \phi(x, Iy)$, $Q(g) \phi : (x, Iy) \mapsto \phi(x, Iyg)$ for $\phi \in V$, is an object of $\mathcal{M}(\mathcal{X})$, that we'll call the vacuum object.

Note that the above objects, which exist for any finite crossed module $\mathcal{X}$, need not be distinct. For example, in the category $\mathcal{M}(\mathcal{R}G)$ (see Example 1) one has $\underline{\mathbf{0}} = \underline{\mathbf{R}}$, while in $\mathcal{M}(\mathcal{D}G)$ one has $\underline{\mathbf{0}} = \underline{\mathbf{1}}$.

Given an object (V, P, Q) of $\mathcal{M}(\mathcal{X})$, a linear subspace $W < V$ is invariant if $P(m)W \subset W$ and $Q(g)W \subset W$ for all $m \in \mathcal{X}_2$ and $g \in \mathcal{X}_1$. An object (V, P, Q) is reducible if it has a nontrivial invariant subspace, otherwise it is irreducible. For a finite crossed module $\mathcal{X}$ there are only finitely many isomorphism classes of irreducible objects in $\mathcal{M}(\mathcal{X})$, which follows from the following generalization of Burnside's classical theorem [13, 15]:

$$\sum_{p \in \operatorname{Irr}(\mathcal{X})} d_p^2 = |\mathcal{X}_1| |\mathcal{X}_2|, \tag{6}$$

where we denote by $\operatorname{Irr}(\mathcal{X})$ the set of (isomorphism classes of) irreducible objects of $\mathcal{M}(\mathcal{X})$, and d_p denotes the dimension of the irreducible $p \in \operatorname{Irr}(\mathcal{X})$.

The notion of direct sum of objects of $\mathscr{M}(\mathcal{X})$ is the obvious one:

$$(V_1, P_1, Q_1) \oplus (V_2, P_2, Q_2) = (V_1 \oplus V_2, P_1 \oplus P_2, Q_1 \oplus Q_2). \quad (7)$$

The analogue of Maschke's theorem states that, for a finite crossed module $\mathcal{X}$, any object of $\mathscr{M}(\mathcal{X})$ decomposes uniquely (up to ordering) into a direct sum of irreducible objects.

The tensor product of the objects (V_1, P_1, Q_1) and (V_2, P_2, Q_2) is the triple $(V_1 \otimes V_2, P_{12}, Q_{12})$, where $P_{12} : m \mapsto \sum_{n \in \mathcal{X}_2} P_1(n) \otimes P_2(n^{-1}m)$ and $Q_{12} : g \mapsto Q_1(g) \otimes Q_2(g)$. The category $\mathscr{M}(\mathcal{X})$ may be shown to be a monoidal tensor category, which in general fails to be symmetric, but it is always braided, the braiding being provided by the map

$$R_{12} : V_1 \otimes V_2 \to V_2 \otimes V_1$$
$$v_1 \otimes v_2 \mapsto \sum_{m \in \mathcal{X}_2} Q_2(\partial m) v_2 \otimes P_1(m) v_1$$

At this point it is worthwhile to take a look the category $\mathscr{M}(\mathcal{X})$ for the two canonical examples of crossed modules considered in Section 2, namely $\mathcal{R}G$ and $\mathcal{D}G$ for a finite group G. In the first case, since $\mathcal{X}_2 = \mathbf{1}$, the map $P : \mathcal{X}_2 \to \mathrm{End}(V)$ is trivial: $P(m) = \delta(m, 1)\,\mathbf{id}$, while the map $Q : \mathcal{X}_1 \to \mathrm{Aut}(V)$ provides a representation of the finite group $\mathcal{X}_1 = G$. Thus, for $\mathcal{X} = \mathcal{R}G$ the category $\mathscr{M}(\mathcal{X})$ is nothing but the category of representations of the finite group G. On the other hand, for $\mathcal{X} = \mathcal{D}G$ the map P is no longer trivial, and a little thought reveals that in this case $\mathscr{M}(\mathcal{X})$ is just the module category of the (Drinfeld) double of the finite group G [2, 3, 8]. It is known that this last tensor category is modular, and describes the properties of the so-called holomorphic G-orbifold models [9]. So, from this point of view, the category $\mathscr{M}(\mathcal{X})$ may be viewed as a common generalization of the module categories of a finite group and of its double.

4. Characters

The notion of group characters is an extremely powerful tool in the study of group representations [13]. Not only do characters distinguish inequivalent representations, but they prove invaluable in actual computations, e.g. the decomposition into irreducibles, the computation of tensor products, etc. As it turns out, a close analogue of group characters exists for the (isomorphism classes of) objects of $\mathscr{M}(\mathcal{X})$. Namely, the character of an object (V, P, Q) of $\mathscr{M}(\mathcal{X})$ is the complex valued function $\psi : \mathcal{X}_2 \times \mathcal{X}_1 \to \mathbb{C}$ given by

$$\psi(m, g) = \mathrm{Tr}_V(P(m) Q(g)). \quad (8)$$

Clearly, characters of isomorphic objects are equal, and it follows from the orthogonality relations to be presented a bit later that characters distinguish inequivalent objects of $\mathscr{M}(\mathcal{X})$. The character ψ of an object of $\mathscr{M}(\mathcal{X})$ is a class function of the crossed module $\mathcal{X}$, i.e. a complex valued function $\psi :$ $\mathcal{X}_2 \times \mathcal{X}_1 \to \mathbb{C}$ that satisfies

(1) $\psi(m, g) = 0$ unless $m^g = m$, for $m \in \mathcal{X}_2$ and $g \in \mathcal{X}_1$;
(2) $\psi(m^h, h^{-1}gh) = \psi(m, g)$ for all $m \in \mathcal{X}_2$ and $g, h \in \mathcal{X}_1$.

The set of class functions of a finite crossed module $\mathcal{X}$ form a finite dimensional linear space $\mathscr{Cl}(\mathcal{X})$, which carries the natural scalar product

$$\langle \psi_1, \psi_2 \rangle = \frac{1}{|\mathcal{X}_1|} \sum_{m \in \mathcal{X}_2, g \in \mathcal{X}_1} \overline{\psi_1(m, g)} \psi_2(m, g), \tag{9}$$

where $\psi_1, \psi_2 \in \mathscr{Cl}(\mathcal{X})$, and the bar denotes complex conjugation[1].

Characters behave well under direct sums and tensor products: the character of a direct sum is just the (pointwise) sum of the characters of the summands, while the character of a tensor product is given by the formula

$$\psi_{A \otimes B}(m, g) = \sum_{n \in \mathcal{X}_2} \psi_A(n, g) \psi_B\left(n^{-1}m, g\right), \tag{10}$$

if ψ_A, ψ_B are the characters of the factors.

Irreducible characters, i.e. the characters of the irreducible objects of $\mathscr{M}(\mathcal{X})$, play a distinguished role, since any character may be written (uniquely) as a linear combination of irreducible ones with non-negative integer coefficients. The basic result about irreducible characters is the following analogue of the generalized orthogonality relations for group characters [13, 15]:

$$\frac{1}{|\mathcal{X}_1|} \sum_{h \in \mathcal{X}_1} \psi_p(m, h) \psi_q\left(m, h^{-1}g\right) = \frac{1}{d_p} \delta_{pq} \psi_p(m, g) \tag{11}$$

for $p, q \in \mathrm{Irr}(\mathcal{X})$, where

$$d_p = \sum_{m \in \mathcal{X}_2} \psi_p(m, 1) \tag{12}$$

denotes the dimension of the irreducible p. From this one can deduce at once that the characters of the irreducible representations form an orthonormal basis in the space $\mathscr{Cl}(\mathcal{X})$ of class functions, and that they also satisfy the second orthogonality relations

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1 Note that for the character ψ of an object of $\mathscr{M}(\mathcal{X})$ one has $\overline{\psi(m, g)} = \psi\left(m, g^{-1}\right)$.

$$\sum_{p \in \mathrm{Irr}(\mathcal{X})} \psi_p(m, g)\, \psi_p(n, h) = \sum_{z \in \mathcal{X}_1} \delta\left(n, m^z\right) \delta\left(h^{-1}, z^{-1} g z\right). \qquad (13)$$

Note that the irreducible characters ψ_p may be computed explicitly for any finite crossed module $\mathcal{X}$, e.g. one has $\psi_{\underline{1}}(m, g) = \delta(m, 1)$ for the identity object $\underline{1}$ of $\mathscr{M}(\mathcal{X})$ (cf. Example 3).

Using the orthogonality relations, one may express the fusion rule coefficient N_{pq}^r, i.e. the multiplicity of the irreducible $r \in \mathrm{Irr}(\mathcal{X})$ in the tensor product of the irreducibles p and q, through the formula

$$N_{pq}^r = \frac{1}{|\mathcal{X}_1|} \sum_{m,n \in \mathcal{X}_2} \sum_{g \in \mathcal{X}_1} \psi_p(m, g)\, \psi_q(n, g)\, \overline{\psi_r(mn, g)}. \qquad (14)$$

To each irreducible $p \in \mathrm{Irr}(\mathcal{X})$ one may associate the complex number

$$\omega_p = \frac{1}{d_p} \sum_{m \in \mathcal{X}_2} \psi_p(m, \partial m), \qquad (15)$$

(remember that d_p denotes the dimension of the irreducible p), which turns out to be a root of unity (of order dividing the exponent of $I = \mathrm{im}\, \partial$), and one may show that[2]

$$\psi_p(m, g\, \partial m) = \omega_p \psi_p(m, g), \qquad (16)$$

for all $m \in \mathcal{X}_2, g \in \mathcal{X}_1$. Combined with the orthogonality relations Eq.(11), this leads to (remember that $K = \ker \partial$)

$$\sum_{p \in \mathrm{Irr}(\mathcal{X})} d_p^2 \omega_p^{-1} = |\mathcal{X}_1| \, |K|, \qquad (17)$$

to be compared with Eq.(6).

To conclude, let's just note that the close analogy with ordinary group characters goes much further, e.g. one may introduce the Frobenius-Schur indicator

$$\nu_p = \frac{1}{|\mathcal{X}_1|} \sum_{m \in \mathcal{X}_2, g \in \mathcal{X}_1} \delta\left(m^g, m^{-1}\right) \psi_p\left(m, g^2\right). \qquad (18)$$

of the irreducible character ψ_p, and show that ν_p may take only the values 0 and ± 1, in perfect parallel with the classical case [13]. Of course, this is related to the fact that ordinary characters of the finite group G are nothing but the characters of the crossed module $\mathcal{R}G$ of Example 1: from this perspective, ordinary character theory of groups is just a special case of the more general theory presented in this note.

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2 This follows from Schur's lemma, upon noting that $\sum_{m \in \mathcal{X}_2} P(m)\, Q(\partial m)$ commutes with $P(n)\, Q(g)$ for all $(n, g) \in \mathcal{X}_2 \times \mathcal{X}_1$.

5. The S matrix and the structure of the vacuum

Up to now, we have seen the close parallel between the structure of the category $\mathscr{M}(\mathcal{X})$ and the representation category of a finite group. We now turn to describe the premodular structure, related to the existence of the so-called S matrix. This is a square matrix, with rows and columns labeled by the irreducibles of $\mathscr{M}(\mathcal{X})$, and with matrix elements

$$S_{pq} = \frac{1}{|\mathcal{X}|} \sum_{m,n \in \mathcal{X}_2} \overline{\psi_p(m, \partial n) \psi_q(n, \partial m)} \tag{19}$$

for $p, q \in \mathrm{Irr}(\mathcal{X})$, where $|\mathcal{X}| = |\mathcal{X}_2| |C| = |K| |\mathcal{X}_1|$ (remember Eq.(1)). This matrix is obviously symmetric, and a simple computation shows that

$$S_{\underline{1}p} = \frac{d_p}{|\mathcal{X}|} > 0, \tag{20}$$

where $\underline{1}$ denotes the identity object of $\mathscr{M}(\mathcal{X})$ (cf. Example 3). The definition of S is motivated by the case of group doubles $\mathcal{D}G$, when it describes the modular properties of the corresponding holomorphic G-orbifolds.

A most important feature of the above S matrix is its relation to the fusion rule coefficients N_{pq}^r appearing in Eq.(14), for one may show that

$$\sum_{r \in \mathrm{Irr}(\mathcal{X})} N_{pq}^r S_{rs} = \frac{S_{ps} S_{qs}}{S_{\underline{1}s}} \tag{21}$$

holds, which is an avatar of Verlinde's celebrated formula [17]. A closely related result states that

$$\sum_{r \in \mathrm{Irr}(\mathcal{X})} N_{pq}^r \omega_r^{-1} S_{\underline{1}r} = \omega_p^{-1} \omega_q^{-1} S_{pq}, \tag{22}$$

where the roots of unity ω_p are given by Eq.(15). But this is not the end of the story since, upon introducing the diagonal matrix $T_{pq} = \omega_p \delta_{pq}$, one may show that

$$STS = T^{-1}ST^{-1}. \tag{23}$$

Should S satisfy the relation $S^4 = 1$, Eq.(23) would mean that the matrices S and T give a finite dimensional representation of the modular group $\mathrm{SL}_2(\mathbb{Z})$, which conforms with Verlinde's theorem [14, 17], i.e.

(1) T is diagonal and of finite order;
(2) S is symmetric;
(3) Verlinde's formula Eq.(21) holds.

Should this be the case, $\mathcal{M}(\mathcal{X})$ would be a Modular Tensor Category. As it turns out, in general this is not the case, because the matrix S of Eq.(19) does only satisfy the weaker property

$$S^8 = S^4. \tag{24}$$

This means that S is not necessarily invertible: it might have a nontrivial kernel. This is the extent to which $\mathcal{M}(\mathcal{X})$ fails to be modular in general.

The lack of invertibility of S is related to the reducibility of the vacuum object $\underline{\mathbf{0}}$ (cf. Example 5). Denoting by μ_p the multiplicity of the irreducible p in $\underline{\mathbf{0}}$, and by $D = |C||K|$ the dimension of $\underline{\mathbf{0}}$, one may show that

$$\mu_p = D \left[S^2 \right]_{\mathbf{1}p,} \tag{25}$$

and that $\mu_p > 0$ if and only if there exists an α such that

$$S_{pq} = \alpha S_{\mathbf{1}q} \quad \text{for all} \quad q \in \mathrm{Irr}\,(\mathcal{X})\,, \tag{26}$$

in which case $\alpha = \mu_p = d_p$ and $\omega_p = 1$. In other words, the irreducible objects of $\mathcal{M}(\mathcal{X})$ that satisfy Eq.(26) for some constant α are precisely the irreducible constituents of the vacuum $\underline{\mathbf{0}}$. The invertibility of S requires that the only such object is the identity $\underline{\mathbf{1}}$, and this condition may be shown to be equivalent to the bijectivity of the boundary map ∂, which in turn is equivalent to $\mathcal{X}$ being isomorphic to $\mathcal{D}G$ for some finite group G. Note also that for $\mathcal{X} = \mathcal{R}G$ every irreducible of $\mathcal{M}(\mathcal{X})$ satisfies Eq.(26), since in this case $\underline{\mathbf{0}} = \underline{\mathbf{R}}$.

Finally, we note that while $\mathcal{M}(\mathcal{X})$ fails to be modular in case ∂ is not bijective, it can nevertheless be turned into an MTC! Indeed, according to the modularizability criterion of Bruguieres [6], one can associate a well-defined MTC (unique up to isomorphism) to any premodular category in which Eq.(26) implies $\omega_p = 1$ and $\alpha = d_p$. But we won't pursue this line any further in the present note, and leave the construction of the corresponding MTC to some future work.

6. Discussion

As we have sketched in the previous sections, to any finite crossed module $\mathcal{X}$ one may associate a premodular category $\mathcal{M}(\mathcal{X})$. In special instances this construction gives back the module category of a finite group or that of its (Drinfeld) double, but in general one gets new premodular categories, which are very close to being modular: they satisfy the modularizability criterion of [6], i.e. they can be turned into a Modular Tensor Category. This opens the way to the construction of a huge number of Modular Tensor Categories starting from (relatively) simple algebraic structures.

As stressed before, the category $\mathscr{M}(\mathcal{X})$ may be viewed as a generalization of the module category of the double of a finite group G, which describes the properties of holomorphic G-orbifolds [2, 3, 8]. This leads to the speculation that for a general crossed module $\mathcal{X}$ the category $\mathscr{M}(\mathcal{X})$, or more precisely its modularisation, should describe the properties of some 'generalized' holomorphic orbifold related to $\mathcal{X}$. To find out whether this vague idea may be made to work seems to be a rewarding task.

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