Polyhedron Models

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Magnus J. Wenninger
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In grateful memory of T. G. who tutored me in the Philosophy of Mathematics

Mathematics possesses not only truth but supreme beauty, a beauty cold and austere, like that of sculpture, sublimely pure and capable of a stern perfection, such as only the greatest art can show.

Bertrand Russell

Contents

Preface to 1978 reprint

The study of polyhedra is one area of mathematics in which the ordinarily abstract and speculative considerations of the subject find very pleasing and attractive visual applications. It is also an area in which both the amateur and the expert in mathematics can work with equal delight.

The enthusiastic response which greeted the first publication of this book (1971) provided ample evidence of this fact, as well as its appearance in a paperback edition (1974), reprinted in a hardcover edition (1975, 1976) and now in still another reprint (1978).

A book which contains as many geometrical drawings as this one, all of which demanded careful draughtsmanship in the originals, could hardly have been printed without errors in their reproduction. Every effort was made to correct these where feasible in the reprinted editions. Further corrections are being incorporated in this present edition.

A word of caution is in place here for the dedicated model maker. Although the original intention of the book was to provide patterns that can be traced from the book and thus used directly in making the models, you may find that for best results very careful workmanship still demands that you make your own full scale drawings of all facial planes from which the patterns or nets are derived.

For the beginner or the inexperienced this has great educational value. For those who already possess the required expertise this will avoid their being led astray by the book. On the other hand in all cases it must be remembered that a model is a model. The full delightfulness of any polyhedron model must ultimately be a matter of intellectual insight.

It may be of interest for readers to know that a definitive enumeration of uniform polyhedra has now been made. John Skilling of the Department of Applied Mathematics and Theoretical Physics at the University of Cambridge has shown that 'the list of Coxeter *et al.* is indeed complete as regards uniform polyhedra in which only two faces meet at any edge. The natural generalization that any even number of faces may meet at an edge allows just one extra polyhedron to be included in the set' (John Skilling, 'The complete set of uniform polyhedra,' *Phil. Trans*. Royal Society of London. Series A, vol. 278, no. 1278).

M.J.W.

January 1978

Preface

This book presents a well-defined set of geometrical solids, the seventy-five (known) uniform polyhedra, together with a representative set of stellated forms. A description of the underlying theory of polyhedra is included to bring out the relationships that exist between the various solids. But mainly this book is simply a set of instructions on how to make models of these solids.

The sources in which you can find an account of the mathematical theory of this topic are given at the end of the book. If in the past you found the study of geometry a bit difficult, or if at present you are not particularly attracted by geometry, you may wonder if this topic will hold your interest. The fact is that you really do not need to understand all the theoretical mathematics involved in the original discovery and classification of these solids. On the other hand you cannot avoid all the mathematics, especially the terminology used here and some of the symbolism.

The objective in this book will be to set down an explanation of the solids, at once simple and practical and not too speculative, one sufficient for the purposes of constructing the models. It is really surprising how much enlightenment will come, following the construction of the models rather than preceding it, and once you begin making them you may find that your enthusiasm will grow. You will soon see that each of these solids has a beauty of form that appeals to the eye in much the same way that the abstract mathematics appeals to the mind of a mathematician.

You may find the number of models presented here overwhelming, some of them extremely complex. Why should anyone want to make them? Maybe the answer is to be found in the reply of a mountain climber when he was asked, 'Why do you want to climb the Matterhorn?' 'The mountain is there, isn't it?' There is another question many people ask when they see these polyhedron models: 'What do you use them for?' Maybe the answer to this is best given by a return question: 'Does beauty need to have uses?' Admittedly the only use a model has, once it has been constructed, is for display purposes. You can make some very attractive mobile models, and generally the constructions make lovely mantelpieces or centrepieces for tables at a banquet on special occasions. Stars seem to go with Christmas and here you have many star forms to choose from.

But on a more technical level you may have seen polyhedron forms used for space satellites. Then again the geodesic dome is found in architecture and in engineering projects. Perhaps the polyhedral forms presented in this book have never been used simply because they have never been widely known.

I have myself constructed all the models presented here and shown in the photographs. How long did it take me? My interest in this topic began in 1958 with a summer course at Columbia Teachers College in New York. During the following year I made my first set of models, those given in section I of this book. My main source was *Mathematical recreations and essays* by Coxeter and Ball. Then between 1959 and 1961 I made all those in *Mathematical models* by Cundy and Rollett. Next I tackled *The fifty-nine icosahedra* by Coxeter, Du Val, Flather, and Petrie. I succeeded in working out my own nets for each of these. The set graced the back wall of my mathematics classroom, growing as I completed each one between 1961 and 1963. The average working time spent on each was about eight hours, plus three or four hours each to discover suitable nets. On the completion of this project I wrote to Professor Coxeter asking about *Uniform polyhedra*. He kindly sent me a complimentary copy, one of three he still had in his possession. This monograph is a detailed account of the mathematical theory of uniform polyhedra. But for the purposes of making the models I inspected the drawings, done by J. C. P. Miller and collected together at the end of the monograph, to discover the facial planes from which I derived the parts. These facial planes are now being presented in this book. A set of photographs was also given in the monograph; these show wire models made by M. S. Longuet-Higgins, but they sometimes represent more than one polyhedron, so they are not the same thing at all as the models presented here.

My working time on the non-convex uniform polyhedron models varied greatly. The simpler ones took three or four hours each, the average would be near eight or ten hours each, a few of the complex models took twenty or thirty hours work. Two of the non-convex snubs required more than one hundred hours work each. Now that the work is complete, I must admit I myself am amazed. But the Chinese proverb applies: If you want to make a journey of a thousand miles, you begin by taking the first step. One step leads to the next, and soon the beauty of the countryside makes you forget the toil of the road.

A special word of thanks is extended to Mr R. Buckley for his truly remarkable calculations on the snub polyhedra and for his astoundingly detailed drawings of their facial planes. Without his help the book could never have been completed. Also a word of thanks to Dr H. Martyn Cundy for his deep interest in the book at all stages of its preparation, and to H. S. M. Coxeter, J. C. P. Miller and M. Longuet-Higgins, who did the original research for *Uniform polyhedra*, and who have provided the source of inspiration from which this book springs. Each in turn provided further encouragement and help to complete the task. Thanks are also due to Stanley Toogood for the photography, to the Syndics of Cambridge University Press for accepting the book for publication, and to the editorial staff of the Press who in an admirable way met the challenge of producing it.

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Foreword

Interest in polyhedra runs through the whole gamut of intellectual activity from the two-year-old child who plays with wooden cubes to the mature mathematician who studies the subtleties of Branko Grünbaum's *Convex polytopes* (Wiley, New York, 1967). Some of the regular and semi-regular solids occur in nature as crystals, others as viruses (revealed by the electron microscope). Bees made hexagonal honeycombs long before man existed, and in human history the making of flat-faced solids (such as pyramids) is as ancient as any other kind of sculpture. The five regular solids were studied by Theætetus, Plato, Euclid, Hypsicles, and Pappus.

A considerable portion of the present book is devoted to 'uniform' polyhedra, which have the same arrangement of regular polygons at every corner. (Such a polyhedron is 'regular' if the polygons are all alike.) In any convex solid, a theorem of Euclid tells us that the angles at a corner must add up to less than 360°. After making a few models for himself, the reader will soon discover that the amount by which the angle-sum falls short of 360° is quite considerable when there are few corners (e.g. 90° for the cube, which has eight corners) but much smaller when there are many (e.g. 12° for the snub dodecahedron, which has sixty corners). This observation was fashioned into a theorem by René Descartes (1596–1650), who proved that the angular defect, added up for all the corners, always makes a total of 720°.

At about the same time, Johann Kepler (1571–1630) wrote an essay on *The six-cornered snowflake* (English edition, Oxford, 1966), in which he revealed his fondness for these figures by remarking (p. 37): 'Now among the regular solids, the first, the firstborn and father of all the rest, is the cube, and his wife, so to speak, is the octahedron, which has as many corners as the cube has faces.' It was Kepler who first published a complete list of the thirteen Archimedean solids, giving them the names by which they are still known. (The work of Archimedes himself had been lost, presumably in the great fire of Alexandria, which was so poignantly dramatized by Bernard Shaw in *Caesar and Cleopatra*.) Kepler also proposed the problem of enumerating the *isozonohedra* (convex polyhedra whose faces are congruent rhombi), and partially solved it by discovering the (first) rhombic dodecahedron and the triacontahedron. But his most important contribution to the ideas of the present book was his proposal to consider non-convex polyhedra whose faces are stellated polygons such as the pentagram (fig. 21). He was probably unaware of the earlier work on stellated polygons by Thomas Bradwardine (1290–1349), who became Archbishop of Canterbury for the last month of his life.

Salisbury Cathedral is such a magnificent building, full of interesting relics, that many visitors fail to notice the tomb of Thomas Gorges, who died in 1610. The stone-carved decorations include a dodecahedron, three icosahedra, and two cuboctahedra, all 'skeletal' in the style of Leonardo da Vinci (1452–1519) who had made skeletal models of many uniform polyhedra using rods to represent the edges. A few miles to the south-west is the pleasant village of Wimbourne St Giles, where Antony Ashley was buried in 1627. His tomb is embellished with a truncated icosahedron, not skeletal but a solid looking just like the author's model **9**.

Since the time of Descartes, many other great mathematicians have contributed to this subject. Euler discovered and proved the famous formula

$$V - E + F = 2$$

which connects the numbers of vertices, edges, and faces of any convex polyhedron. Gauss used an irregular spherical pentagram (his *pentagramma mirificum*) to explain Napier's rules in spherical trigonometry. Cauchy proved that every convex polyhedron with rigid faces and hinged edges is rigid. Hamilton invented the Icosian Game (W. W. Rouse Ball, *Mathematical recreations and essays*, Macmillan, 1967, p. 262).

Von Staudt gave a new proof for Euler's formula. Schläfli extended the theory to n dimensions. Klein wrote a highly influential book called *Lectures on the icosahedron*. Fedorov returned to Kepler's problem of isozonohedra, discovering a strangely oblate-looking rhombic icosahedron; and Bilinski (as recently as 1960) completed the enumeration by discovering a second rhombic dodecahedron which would fit snugly into a box of unit height, breadth τ and length τ^2, where τ is the number $(\sqrt{5}+1)/2$ which belongs to the celebrated 'divine proportion' or 'golden section'.

In his infectiously enthusiastic style, the author gives clear instructions for making models of many kinds of polyhedra. These instructions are illustrated by photographs of his own collection, including what is almost certainly the only complete set ever made of the known uniform polyhedra. But photographs cannot really show the models in their full splendour. The most complicated 'snub' solids are not only extremely difficult to make but also highly decorative: a perfect instance of the connection between truth and beauty.

H.S.M.C.