Introduction: uniform polyhedra

If you are being introduced to this topic for the first time, your first question might well be 'What is a polyhedron?' You may recall that geometry itself is sometimes (not too exactly) defined as the study of space or of figures in space—two dimensional for plane geometry and three for solid geometry. The idea of sets is perhaps familiar also. If you use the language of sets, a plane figure may be defined as a set of line segments enclosing a portion of two-dimensional space. Such a plane figure is called a polygon. A polyhedron is then defined as a set of plane figures enclosing a portion of three-dimensional space.

All the terms used in this subject are derived from classical Greek. Plato, the famous Greek philosopher, left the imprint of this thought deeply fixed in Euclid's *Elements*. This ancient book, for centuries the only textbook of geometry, was concerned with 'ideal' lines and 'ideal' figures. The ideal lines are straight and the ideal polygons are regular, that is, they have all sides and all angles equal. The simplest regular polygon is the equilateral triangle. It is the simplest because it has the least number of line segments possible to enclose a portion of two-dimensional space. It is an interesting fact that Euclid's *Elements* begins with a proposition describing how to construct an equilateral triangle and ends with a study of the five regular solids. Each of these has regular polygons of the same kind for all its faces. They are known today as the five Platonic solids. The tetrahedron, which has four equilateral triangles for its faces, is the three-dimensional analogue of the two-dimensional equilateral triangle. It is the simplest polyhedron, since it has the least number of faces possible to enclose a portion of three-dimensional space.

With the equilateral triangle the following polygons enter the picture: the square (four sides), the pentagon (five sides), the hexagon (six sides), the octagon (eight sides) and the decagon (ten sides), all of course only as regular polygons. Once you begin to make the models described in this book, you will quickly learn to draw all these figures accurately and will become acquainted with important properties belonging to each, especially the number of degrees in each interior angle. Not all the regular polygons are to be found in the regular solids; in fact only three are used. The hexahedron (six faces), commonly called the cube, has squares; the octahedron (eight faces) again has equilateral triangles; the dodecahedron (twelve faces) has all pentagons; and finally the icosahedron (twenty faces) has twenty equilateral triangles. Euclid's *Elements* closes with a proof that there are only five regular polyhedra.

A little experimenting with cardboard figures will soon lead you to see the reasoning behind a formal proof. Just as in a polygon two sides meet at a point called a vertex of the figure, so in a polyhedron two faces meet at or on a line (or in a line—the mode of expression is variable). Thus each face shares one of its sides as a line in common with another face. These lines are called the edges of the polyhedron. So each edge of a polyhedron belongs to exactly two faces and no more. The edges all meet at a point called a vertex of the polyhedron.

In the tetrahedron three edges meet at each vertex, or to put it another way, each vertex is surrounded by three triangles. It is enlightening to lay out these three triangles flat and to notice the sum total of the number of degrees in the angles at a common vertex. Three sixties give 180 degrees. If a fourth triangle is introduced the total is 240 degrees, but now you have a vertex of the octahedron. Introducing a fifth triangle gives 300 degrees, and you have a vertex of the icosahedron. A sixth triangle gives 360 degrees and you can see immediately that no polyhedral vertex arises. Everything stays flat.

Next you can try squares. A minimum of three is required, three 90s give 270 degrees, and a vertex of the cube can be formed. Adding a fourth square brings the total to 360 degrees and again you are left—flat. With pentagons the minimum of three will give you a vertex of the dodecahedron; four are too many, the total going beyond 360 degrees. With hexagons the minimum of three is already too many, three times 120 degrees. So no regular polyhedron exists with only hexagons for faces. And similarly for polygons with any greater number of sides. In this way you can see that only five regular solids are possible.

There is another set of solids known as the Archimedean or semi-regular solids. These all have regular polygons as faces and all vertices equal but admit a variety of such polygons in one solid. There are thirteen such solids and they are ascribed to Archimedes because he first enumerated them, although his work on them has been lost. References to his work on this subject are found in the writings of Pappus, a mathematician of the third century A.D. Kepler was the first of the moderns to formulate a complete theory concerning them.

The Archimedeans can be broken down into various subsets. There are first of all the five derived by the process of truncation from the five Platonic solids. To truncate literally means to cut off. Truncation thus implies the removal of some portion of a solid, actually the removal of a portion near each vertex along with the vertex itself. This can be done to the Platonic solids in such a way that the new faces are again regular polygons while the portions of the former faces that are left also form new regular polygons. For example the tetrahedron can be truncated so that the four triangles become four hexagons and the new faces are four new triangles. Five Archimedeans are thus generated. They are named simply: the truncated tetrahedron, the truncated hexahedron (cube), the truncated octahedron, the truncated dodecahedron, and the truncated icosahedron. Another subset, containing only two members, is that known as the quasi-regular polyhedra. The designation 'quasi-' implies that the solid has only two kinds of faces, each face of one kind entirely surrounded by that of the other kind. They are the cuboctahedron and the icosidodecahedron. You will find a fuller treatment of these two later on in this book (see pp. 25 and 26).

Then there are the two called the rhombicuboctahedron and the rhombicosidodecahedron. These two are sometimes named the small rhombicuboctahedron and the small rhombicosidodecahedron to distinguish them from two others called the great rhombicuboctahedron and the great rhombicosidodecahedron. If truncation is applied to the two quasi-regular solids, the cuboctahedron and the icosidodecahedron, the new faces that arise are at best rectangles and thus do not come out as regular polygons. But with some modifications these rectangles can be turned into squares. Because of this some authors refer to the great rhombicuboctahedron and the great rhombicosidodecahedron as the truncated cuboctahedron and the truncated icosidodecahedron. In this book they are named the rhombitruncated cuboctahedron and the rhombitruncated icosidodecahedron. The prefix *rhombi-* implies extra square faces (across edges) of the two quasi-regular solids. With this the designation *small* may be dropped from the names of the former two.

Finally there are two snub versions, one of the cube and one of the dodecahedron. This snub quality gives these a twisted appearance which makes each of them turn out in either of two forms—with a right- or left-handed twist. These mirror image pairs are also called enantiomorphous pairs.

If you are ambitious enough and systematic enough you can also prove to your own satisfaction that the total enumeration of thirteen is complete, that there are no more, by using the same approach here that you used for the five Platonic solids. The appropriate theorem from solid geometry that applies here states that the sum total of the face angles of any convex polyhedral angle is less than 360 degrees. After you have tried all possible combinations of regular polygons for which this theorem remains true you will come up with exactly the thirteen Archimedean solids, and two infinite families, the prism (with square side-faces) and antiprisms (with equilateral triangular side-faces).

(For further details see L. Lines, *Solid geometry*, pp. 159–67.)

The union of these two sets, the Platonic and the Archimedean solids, together with the two infinite sets of prisms and antiprisms, yield the set known as the convex uniform polyhedra. A polygon is convex if no interior angle is greater than 180 degrees. Analogously a polyhedron is convex if no dihedral angle (formed by the intersection of two faces with its vertex on or in an edge) is greater than 180 degrees. Convex is the opposite of concave, bending in on itself. A polyhedron with dimples, dents or grooves in it would be non-convex or concave. The word 'uniform' implies that all faces are regular polygons and all the vertices of the polyhedron are alike. In a uniform polyhedron the polygons around any vertex occur in the same order in every other vertex. For example in the rhombicosidodecahedron the order going around a vertex is: a triangle, a square, a pentagon, and another square. The same holds true at every vertex.

The word 'enantiomorphous' occurs frequently in the following pages. It simply means the property of being right- or left-handed, as in a pair of gloves, or in mirror image pairs. In colour arrangements, if the order around a vertex of a polyhedron is taken in clockwise rotation, the enantiomorphous arrangement will be obtained by taking the same order of colours in counterclockwise rotation.

The following abbreviations will be used for colours: Y yellow, B blue, O orange, R red, G green, W white, T tan.

All this terminology and all these classifications will undoubtedly become clearer to you and more meaningful after you have made the models in section I, the convex uniform polyhedra.

Mathematical classification

This section may be omitted at a first reading.

A uniform polyhedron can be enclosed within a sphere, so that its axes of symmetry pass through the centre of the sphere. By central projection the edges of the polyhedron can then be made to generate a network of arcs of great circles decomposing the surface of the sphere into spherical polygons, one for each face of the polyhedron. The planes of symmetry of the solid will likewise decompose the surface of the sphere into a network of spherical triangles, four for each edge of the solid if it is a Platonic solid. These spherical triangles are called Möbius triangles, because it was Möbius (1849) who first observed this. He illustrated this fact by means of his polyhedral kaleidoscope, consisting of three mirrors forming a trihedral angle. Given certain dihedral angles between these mirrors and given an object to mark a point, the images of the object (together with the object itself) mark the vertices of the polyhedron. Another perhaps easier illustration of Möbius triangles is simply a special set of great circles inscribed with chalk on a slated globe. Some of the intersections of the great circles mark the vertices of the polyhedron. This tessellated network of spherical triangles covers the globe once. All the triangles are congruent.

In symbols one of these triangles can be described by (pqr) where p, q, r are integers and the angles of the triangle are $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$ Here p, q, r can only be 2, 3, 4, or 5. But one or more of p, q, r may be rational; that is, certain fractions may be used as replacements for p, q, r, leading to Schwarz triangles. It was Schwarz (1873) who first listed the possibilities. It has been shown that a set of Schwarz triangles covering the globe more than once but still a finite number of times is equivalent to a set of Möbius triangles. Thus Schwarz triangles may be classified as tetrahedral, octahedral or icosahedral, depending on the Möbius triangles to which they are related. (For further information see: Coxeter *et al.*, *Mathematical recreations and essays*, *Regular polytopes*, and *Phil. Trans.* (1954), **246**A, 401.

These ideas are easier to visualize with the aid of models. You can make your own polyhedral kaleidoscope with three mirrors cut in the shape of circular sectors. The radius must be fairly large, twelve inches or more; the central angles of these sectors must be as follows:

for the tetrahedral kaleidoscope 54° 44′, 54° 44′, 70° 32′;

for the octahedral 35° 16′, 45°, 54° 44′;

for the icosahedral 20° 54′, 31° 43′, 37° 23′.

Interesting as it is to *play* with these mirrors, they are not always so easy to come by, nor are they completely satisfactory. So it is just as good or better to make models of the spherical triangles using the same index card or coloured tag you use in the other models. By repeating these spherical triangles a sufficient number of times you can make a model of the entire sphere as an intersecting set of great circles. In fact the colours can be worked out so that they illustrate the great circles but this calls for more work than is needed in a model all of one colour.

The tetrahedral case is the easiest to begin with. The parts are cut as shown in fig. 1. Score these parts on the radial lines, then fold them forming a model of a spherical triangle. The cementing is done using only one tab as shown. Twenty-four of these are needed and they are simply cemented to each other, flat surface to flat surface, so that the tab joint disappears between the two surfaces. You may do the work in sections. One of these sections has six spherical triangles as shown in fig. 2. The angles are $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}$. Four of these sections complete the model.

For the octahedral case you may follow the same procedure with another set of parts shown in fig. 3. Forty-eight of these are needed, two enantiomorphous sets of twenty-four in each. You may make these any convenient size. In fact

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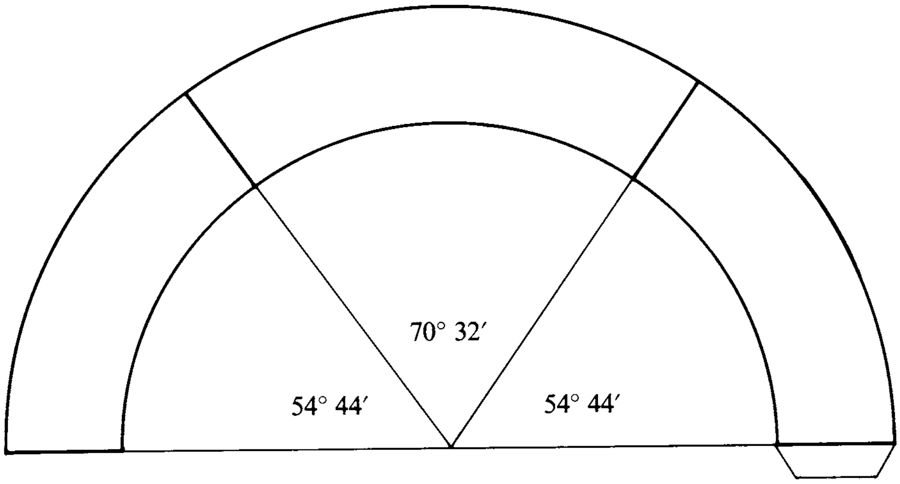

Fig. 1

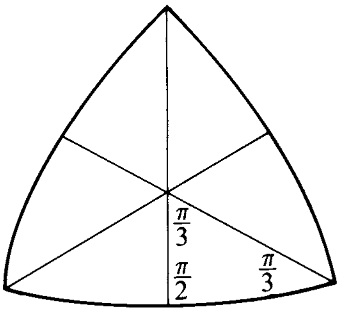

Fig. 2

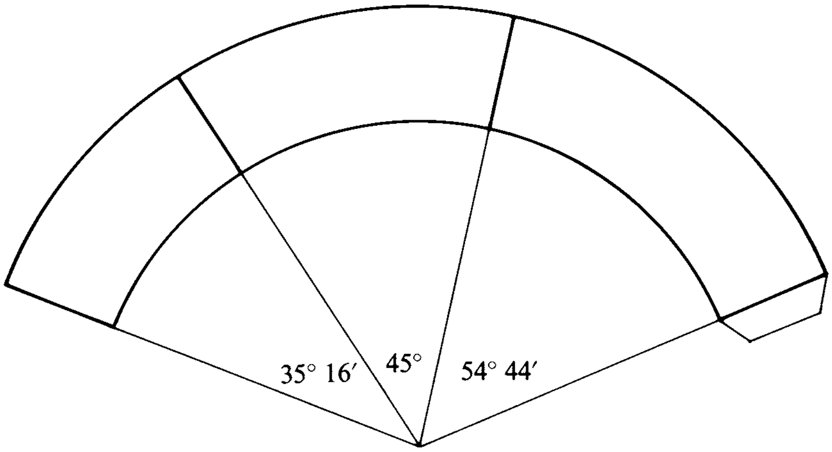

Fig. 3

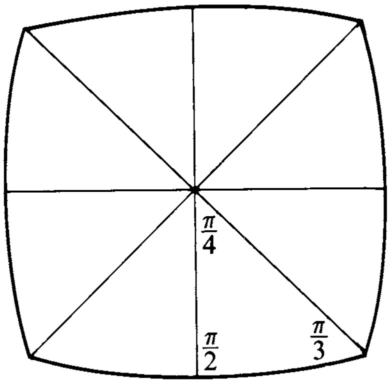

Fig. 4

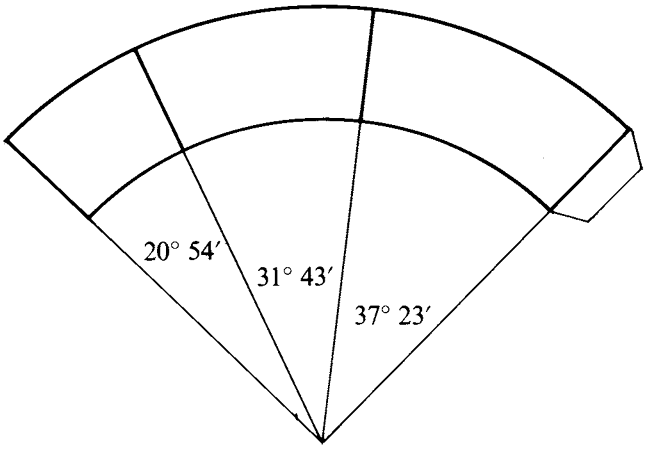

Fig. 5

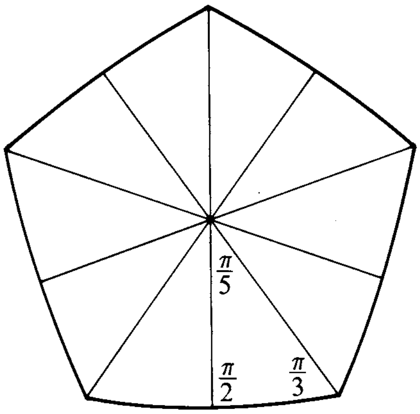

Fig. 6

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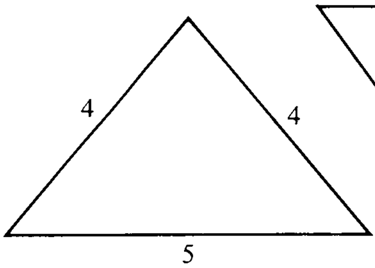


Fig. 7

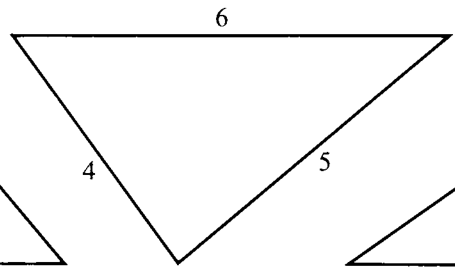


Fig. 8

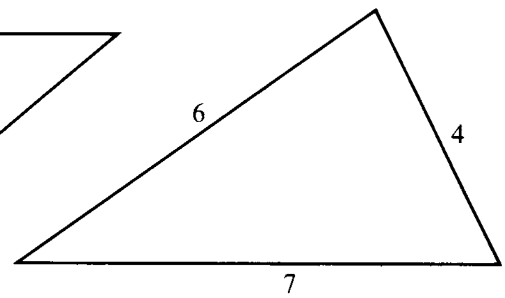


Fig. 9

you may make the circular band wider or narrower as you please, even leaving all the interior, which of course will bring you right to the centre of the sphere. The sections in this case begin to reveal the fact that the octahedron is the dual of the cube, since one of these sections may be the eight spherical triangles arranged as shown in fig. 4. The angles are $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}$. Six of these sections complete the model.

The icosahedral case calls for more work because of the greater number of parts, but the procedure is still the same. It is well worth the effort it takes, because it will bring you a great deal of enlightenment. The openness of the model on the interior has great advantages. One hundred and twenty of these parts are needed, two enantiomorphous sets of sixty in each. The sections are pentagonal, ten spherical triangles to a section. The angles are $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{5}$. Twelve of these sections (see fig. 6) complete the model.

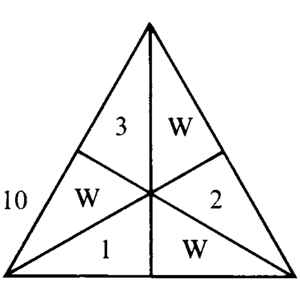
There is still another way to make models illustrating Möbius triangles. It amounts to making a polyhedron whose faces are plane triangles with the same vertices as the spherical triangles. If the sides of a spherical triangle are p, q, r, namely p, q, r are the angles subtended at the centre of the sphere, the corresponding plane triangles have sides proportional to

$$\sin \tfrac{1}{2}p : \sin \tfrac{1}{2}q : \sin \tfrac{1}{2}r.$$

The three cases are shown in figs. 7, 8 and 9 and of course they call for the same number of parts respectively as the spherical triangles to which they correspond. The models may also be done by following the same sectional procedure as for the spherical cases. The numbers are approximate measures in linear units of the sides of these triangles. If you use centimetres you can get satisfactory results.

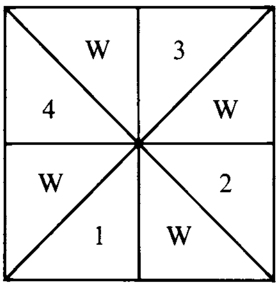
You can also get some striking colour effects by making one set of triangles all W and then the others in the usual colours. The drawings below show the respective sections and their colour tables.

Fig. 10



	1	2	3
(1)	Y	B	O
(2)	Y	R	B
(3)	B	R	O
(4)	O	R	Y

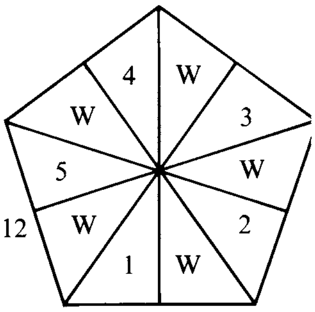
Fig. 11



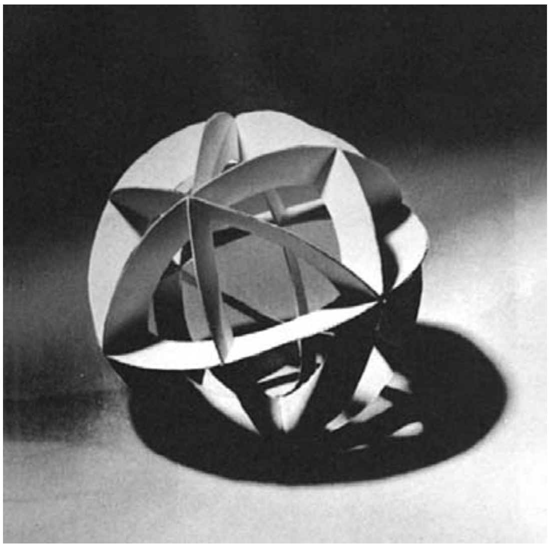
	1	2	3	4
(0)	Y	B	O	R
(1)	O	R	B	Y
(2)	R	Y	O	B

(The other three are enantiomorphic to these.)

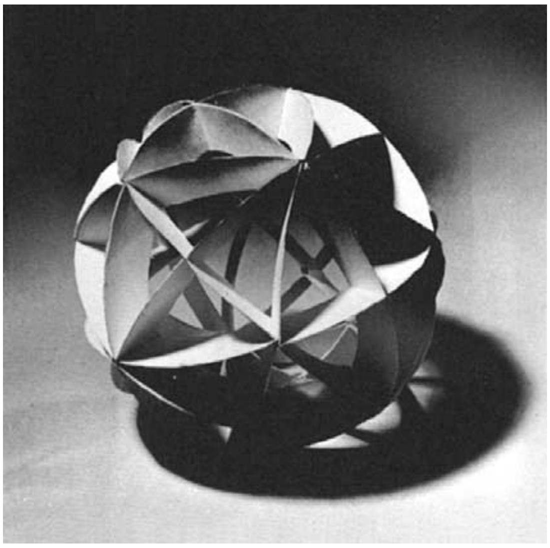
Fig. 12



(See p. 18 for the icosahedral table.)

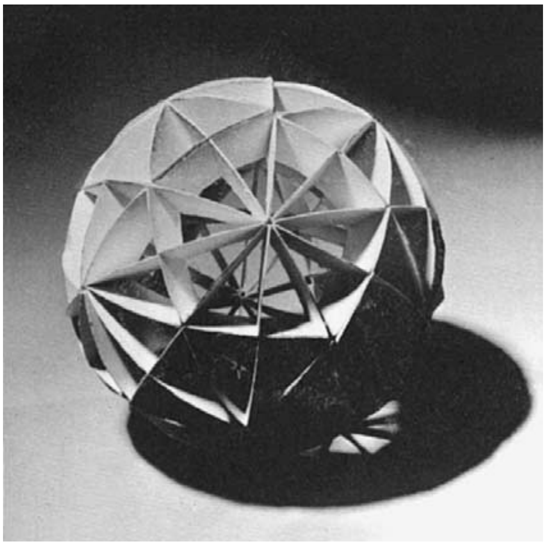


A. Tetrahedral



B. Octahedral

C. Icosahedral



Note that the tetrahedral case has the sections numbered (1), (2), (3), (4). The tetrahedron and the truncated tetrahedron are the only uniform polyhedra that do not have their vertices distributed in diametrically opposite pairs. The (0) section in the other two cases may be taken as the north polar section. Then in the octahedral case (1) and (2) are cemented in place something like the faces of a cube. These are followed by the enantiomorphous arrangement of the same two sections, thus completing the four side-faces of the cube. The enantiomorph of (0) then completes the model. The icosahedral case is the most interesting. You may begin with the (0) section, cementing together ten triangles alternating W with one of each of the colours. Follow the icosahedral colour arrangement as shown on p. 18. Then as you complete each of the other derived, but there is an interchange in the number of faces and vertices. Moreover the kinds of faces and vertices are such that an n-sided face in the original solid yields an n-edged vertex in the dual solid. The three polyhedrons just described thus turn out to be duals of **7**, **15** and **16**, respectively.

Once you have made these models it is a good exercise in spatial imagination to use them, especially the spherical models, to locate the faces, edges and vertices of the convex uniform polyhedra. The diagrams here may serve as a guide. The numbers designate the vertices whose images are the vertices of the polyhedron designated by the same number in the summary, p. 9. The snubs, **17** and **18**, are not indicated below. The vertices of these depend on a suitable point being chosen within the triangles. The exact

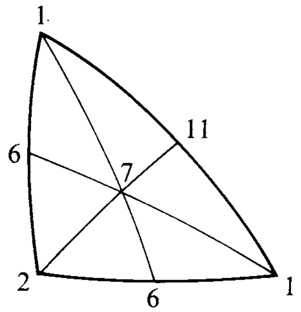


Fig. 13

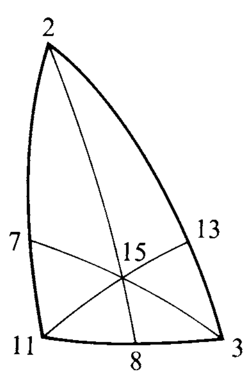


Fig. 14

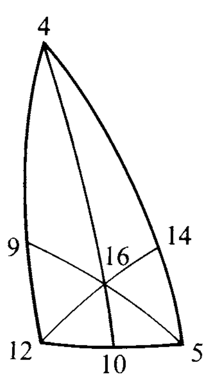


Fig. 15

sections (1), (2), (3), (4), (5), cement it to the (0) section first and then to its neighbour in dodecahedral fashion. The next set of six sections have the enantiomorphous order of colours. They are placed diametrically opposite their counterparts. You will be delighted with the pin-wheel appearance that turns up on all three of these cases. It is most pronounced in the icosahedral case. It is worth mentioning here that these three models are actually Archimedean duals. Dual solids are those which have the same number of edges as the original solids from which they are mathematical details can be found in L. Lines, *Solid geometry*, pp. 175–84. The construction there relates the snubs to the circumscribed cube and dodecahedron, showing how to find the vertices of the snubs on the faces of the cube and dodecahedron. Then by central projection these same points can be located in the spherical triangles.

The summary that follows is an attempt to bring together the various aspects relating principally to the symbols used for each polyhedron. You need not master this material to make the

Summary of convex uniform polyhedra together with their symbols

Regular polygons:

triangle $\{3\}$	square $\{4\}$	pentagon $\{5\}$
hexagon $\{6\}$	octagon $\{8\}$	decagon $\{10\}$

Uniform polyhedra:

Platonic solids (regular solids)

1.	tetrahedron	$\{3, 3\}$	$= 3 \mid 2\ 3$	$= 3^3$
2.	octahedron	$\{3, 4\}$	$= 4 \mid 2\ 3$	$= 3^4$
3.	hexahedron (cube)	$\{4, 3\}$	$= 3 \mid 2\ 4$	$= 4^3$
4.	icosahedron	$\{3, 5\}$	$= 5 \mid 2\ 3$	$= 3^5$
5.	dodecahedron	$\{5, 3\}$	$= 3 \mid 2\ 5$	$= 5^3$

Archimedean solids (semi-regular solids)

6.	truncated tetrahedron	$t\{3, 3\}$	$= 2\ 3 \mid 3$	$= 3.6^2$
7.	truncated octahedron	$t\{3, 4\}$	$= 2\ 4 \mid 3$	$= 4.6^2$
8.	truncated hexahedron	$t\{4, 3\}$	$= 2\ 3 \mid 4$	$= 3.8^2$
9.	truncated icosahedron	$t\{3, 5\}$	$= 2\ 5 \mid 3$	$= 5.6^2$
10.	truncated dodecahedron	$t\{5, 3\}$	$= 2\ 3 \mid 5$	$= 3.10^2$
11.	cuboctahedron (quasi-regular)	$\left\{ {3 \atop 4} \right\}$	$= 2 \mid 3\ 4$	$= (3.4)^2$
12.	icosidodecahedron (quasi-regular)	$\left\{ {3 \atop 5} \right\}$	$= 2 \mid 3\ 5$	$= (3.5)^2$
13.	(small) rhombicuboctahedron	$r\left\{ {3 \atop 4} \right\}$	$= 3\ 4 \mid 2$	$= 3.4^3$
14.	(small) rhombicosidodecahedron	$r\left\{ {3 \atop 5} \right\}$	$= 3\ 5 \mid 2$	$= 3.4.5.4$
15.	rhombitruncated cuboctahedron	$t\left\{ {3 \atop 4} \right\}$	$= 2\ 3\ 4 \mid$	$= 4.6.8$
16.	rhombitruncated icosidodecahedron	$t\left\{ {3 \atop 5} \right\}$	$= 2\ 3\ 5 \mid$	$= 4.6.10$
17.	snub cube	$s\left\{ {3 \atop 4} \right\}$	$= \mid 2\ 3\ 4$	$= 3^4, 4$
18.	snub dodecahedron	$s\left({3 \atop 5} \right)$	$= \mid 2\ 3\ 5$	$= 3^4, 5$

models in this book, but it is interesting to know that the details have been worked out. If you should ever want to undertake further investigation in this field you would have to be thoroughly acquainted with the details.

The Schläfli symbol is given first, then the symbol with dashes, "|" as used in *Uniform polyhedra*, then another symbol as used in *Mathematical models*. In the symbol $\{p, q\}$, p names the polygon that appears in the faces, q names the polygon that appears in the vertex figure. For an explanation of the dashes, see the following page. $\left\{ {p \atop q} \right\}$ simply names the two kinds of polygons found in the faces of the quasi-regular solids. It is an extension of the Schläfli symbol. So too are t, r, s to mean truncated, rhombic and snub respectively. Rhombic implies the existence of extra square faces. Snub implies the existence of extra triangular faces.

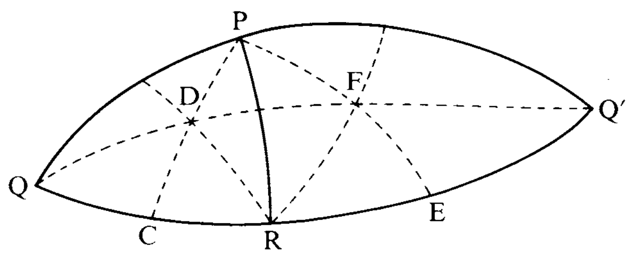


Fig. 16

The meaning of the dashes, "|", may be briefly summarized as follows: A spherical triangle PQR whose angles are $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$ may also be named (pqr). In terms of the polyhedral kaleidoscope:

The polyhedron whose vertices are the images of P is $p|q$ r (or $p|$rq).

$$p|q2 = p|2q = \{q, p\}$$
$$q|p2 = q|2p = \{p, q\}$$
$$2|pq = \begin{Bmatrix} p \\ q \end{Bmatrix}.$$

The polyhedron whose vertices are the images of C is qr$|p$ (or r$q|p$). C is the point of intersection of the bisector PC of the angle QPR with the opposite side QR.

$$pq|2 = \mathrm{r} \begin{Bmatrix} p \\ q \end{Bmatrix}$$
$$2q|p = \mathrm{t}\ \{p, q\}.$$

The polyhedron whose vertices are the images of D is pqr$|$. D is the incentre of the triangle PQR.

$$pq\mathrm{r}\,| = p\mathrm{r}q| = q\,\mathrm{r}\,p| = \text{etc.}$$
$$2pq| = \mathrm{t} \begin{Bmatrix} p \\ q \end{Bmatrix}.$$

E and F apply only to the non-convex uniform polyhedra. They are given here to complete the summary. PE is the external bisector of the angle at P. F is the excentre.

Suppose the angle $Q'PR$ is $\frac{\pi}{p'}$; then $\frac{1}{p}+\frac{1}{p'}=1$. If the angle PRQ is $\frac{1}{2}\pi$, then the polyhedron whose vertices are the images of E is $2q|p'$.

$$p'q|2 = \mathrm{r}' \begin{Bmatrix} p \\ q \end{Bmatrix}$$
$$2q|p' = \mathrm{t}'\{p, q\}.$$

The polyhedron whose vertices are the images of F is $2\,p'q|$.

$$2p'q| = \mathrm{t}' \begin{Bmatrix} p \\ q \end{Bmatrix}.$$

The polyhedron whose vertices are the images of a suitable point within PQR is $|\,pq$r.

$$|2pq = \mathrm{s} \begin{Bmatrix} p \\ q \end{Bmatrix}.$$

The symbols r′ and t′ stand for quasi-rhombic and quasi-truncated respectively.