PART I

1

Introduction

This book is the culmination of more than ten years work by the author on ways of teaching undergraduate students how to become not only competent mathematicians but also skilled users of mathematics in the solution of problems arising in the real world.

In a degree course, at least in mathematics and those related disciplines in which mathematics is a tool, three things must be taught. Firstly, there is a body of factual knowledge and technique which students must acquire. Secondly, the skill of extending their mastery of factual knowledge and technique must also be acquired, that is they must learn how to learn more mathematics as and when the need arises. Thirdly, since all the mathematical knowledge and technique in the world is of little use to the practical mathematician, engineer or scientist if the skill of applying that knowledge to their professional problems is missing, students must learn how to use their mathematical knowledge in solving the problems of the real world. In the past the second and third of these aspects of mathematics have not been formally taught, rather it was assumed that students would acquire them incidentally (one might almost say accidentally) whilst studying the facts, theorems and techniques of mathematics. Indeed there has been considerable doubt as to whether these skills could be taught effectively at all. At the same time, though, one of the classical justifications advanced for teaching mechanics has always been that, through the study of mechanics, students will learn something about the art of applying mathematical theory to the problems of the real world. However, it is now increasingly accepted that, whilst it may be extremely difficult if not impossible to learn these skills merely by attending lectures about them (as indeed it is difficult to learn to ride a bicycle by attending lectures on bicycle riding), they can be more or less successfully learnt by practical activity (as can riding a bicycle).

The author's explorations of means of teaching both the second and third aspects of mathematics mentioned above have resulted in two innovations in his own teaching. Firstly, the use of the 'guided reading' method as a means of helping students to develop the skills of extending their mathematical knowledge has been explored. Work relating to this innovation is reported in UMTC (1978) and Clements and Wright (1983). Secondly, to meet the requirement to teach the skills of applying mathematics to the problems of the real world a series of exercises, termed 'case studies', has been created. It is the aim of this book to explain both the underlying principles of these case studies and how they have been used to help students learn the art of applying their mathematical knowledge to the problems of the real world.

The object of these case study exercises, then, is to give undergraduates some experience of applying their mathematical knowledge to the sort of problems that arise in industry and commerce. Each of these case studies is based on a problem which has, at some time, faced an actual company or organisation. The material on which the studies are based was contributed by those organisations together with an outline of the way in which those originally faced with the problem attempted to solve it. Each chapter in part II of this book presents, in a realistic way, a problem situation facing a mythical organisation in the solution of which mathematical skills may assist. The problems are presented in the form of reconstructed design drawings, correspondence, memoranda and reports. Problems in the real world do not, of course, arise neatly packaged and expressed in mathematical notation. Instead they often arise in messy, confused ways and usually expressed in someone else's terminology (for instance chemical or engineering terminology). This confusion is reflected in the presentation of the case studies via the reports, memoranda and other documents.

Each of the chapters in part II of the book also contains an outline of the approach to solving the problem which was adopted by the original organisation which faced the problem and a commentary on the variations on these approaches which have been produced by student groups studying these problems over the years. It is important to realise that the approach adopted by the donor organisation is not necessarily 'correct' or 'ideal'. In the real world solutions are rarely 'correct' and not often 'ideal'. Most real problems admit of a variety of solutions. The solution method chosen for any given problem must be one that is appropriate under the circumstances prevailing. The relevant circumstances may include non-mathematical constraints as well as mathematical considerations. Problem solving is a much more open-ended activity than theorem proving.

In part I of the book some more general material on mathematical modelling and the use of the Case Study course at Bristol University is presented. It is hoped that this material will assist both lecturers wishing to use the material in part II in their own teaching and students wanting to learn more about applying mathematics to the problems of the real world.

It is appropriate, at this point, to say something more detailed about the ways in which this book might be used. This depends, of course, on who you, the reader, are and on what you hope to learn from the book. Firstly, let us consider what use an individual student of mathematics (and let us not restrict this category to undergraduates but include students of all ages and levels of experience) might make of the book. If you are such a student it would be possible for you merely to read this book much as you would any other book. If you restrict yourself to this however, you will miss much of the benefit that is to be derived from the book, and incidentally much of the pleasure and satisfaction! Ideally you should use this book in association with a structured course in which you will work with others on the solution of the problems in part II of the book. In this way you will enjoy the same benefits that those working in industry and commerce enjoy – the opportunity to discuss with your fellows the advantages and disadvantages of the various possible approaches to the problem, the joint decision taking about which avenue to pursue and the mutual support and stimulus which is derived from common endeavour. It goes without saying, of course, that the benefit you derive from such problem solving will be greatly reduced if you read the outline of the donor's solution before attempting your own. Such prior knowledge will almost certainly constrain your own thinking and implicitly limit your potential creativity. By attempting your own solution of the problem you will be stimulated not only to work towards an appropriate solution but also to consider and evaluate a range of possible ways of obtaining that solution.

If, through force of circumstance, you have to use this book as an isolated individual you will still derive more benefit from thinking out for yourself a solution which answers the problem than you will by merely reading a solution provided by someone else. In many ways it is true to say of this book that the correctness of the answers you obtain is less important than the experience of selecting or devising mathematical ways of answering the problems posed.

Whether you are tackling each problem on your own or as a member of a group, taking a course in an educational institution or reading this book at home for general interest, you will obtain much benefit from the struggle to apply your mathematical knowledge to these novel situations. You should anticipate that this will be, at first, a disorientating and perhaps, to a certain extent, a frightening situation. Do not be disheartened but persevere and you will soon be amply rewarded by the satisfaction of finding yourself able to enhance your understanding of the world in which you live and work through the application of mathematical principles and techniques.

So far we have considered the uses students might make of this book. If, on the other hand, you are a teacher or lecturer seeking stimulation, guidance or just material for a course in the application of mathematics you will find that this book addresses your needs also. Obviously the case studies described in part II of the book can be incorporated into your courses and the outline of the original solution adopted and the commentary on likely or possible student variations can be used in your courses in whatever ways you see fit. In chapter

4 there is an extensive description of the course for which this material was originally developed and within which it has been used successfully for a number of years. Chapters 2 and 3 contain less specific material relating to mathematical modelling in general. In order to understand why this material is included let us return to the analogy between learning to be a competent practitioner of mathematical modelling and learning to ride a bicycle. It was suggested above that these two share the property that neither is easily learnt by attending lectures on the theory of the activity and that both are best learnt by practice. The analogy between bicycle riding and mathematical modelling may be extended to include the role of the teacher. Whilst the learner bicycle rider may only acquire the desired expertise through practice, teachers of bicycle riding will be much more effective if they understand something of the theory of bicycle riding. Such knowledge enables them to analyse the performance of the learner and give appropriate advice and instruction, reinforcing and encouraging appropriate behaviours in the learner and correcting inappropriate ones. This is not to say that those without the theoretical understanding of bicycle riding are not capable of teaching it, but their effectiveness in so doing is reduced by their lack of knowledge. So too with mathematical modelling – experienced practitioners of the art may be capable of passing on their art and skill in an instinctive way but those who wish to be the most effective teachers of the subject must not only have experience of the art and practice of mathematical modelling but must also possess a deeper understanding of the subject and have appropriate mental models of the process which they desire to teach.

Again, whilst practice is the best (and probably the only) way to take the initial steps in learning bicycle riding, once the rider has acquired a minimal practical skill advancement to the highest levels of practical skill in riding is aided by theoretical knowledge of the mechanics and dynamics of bicycles. Perhaps this assertion seems a little fanciful in respect of a relatively simple activity like bicycle riding – if so then translate bicycle riding to driving a motor vehicle. The learner driver acquires his or her initial skill by practical means but, before the driving test can be tackled, some theoretical knowledge is needed to illuminate the best way of using, on the roads, the practical skills which have been acquired. Drivers who wish to advance to passing the test of the Institute of Advanced Motorists, or to reach the even higher standards of the Police traffic patrol driver or the qualified chauffeur, will certainly need further theoretical knowledge of the subject as well as additional practical skill. The analogy carries through to mathematical modelling. After acquiring a basic skill in modelling through practical activities, those who wish to become more expert are helped on the route by some study of the theory of modelling as well as more advanced practice and experience. Such theoretical knowledge helps them to place their practical activities in context and to guard against suboptimal or dysfunctional behaviour.

From this analogy it may be concluded that, at some time or other, both teachers and students of mathematical modelling need to study the theoretical background to, and methodologies of, modelling as well as the practicalities of the subject. To meet these needs chapter two presents an analysis of the background to mathematical modelling, including some comments on the history of the subject and an outline of the development of methodologies of modelling, and chapter three attempts, insofar as such an attempt is possible, to illuminate the practicalities of modelling. This attempt takes the form of something akin to thinking aloud, that is several simple models are developed and comments are made on how and why various decisions were taken during the development process.

In these ways this book attempts to satisfy the needs of both teachers and students of mathematics for assistance in developing those skills which experienced users of mathematics routinely deploy whilst using their mathematical knowledge to illuminate their understanding of the world.

2

The background to mathematical modelling

In chapter one it was suggested that a degree course in mathematics should seek to teach firstly a body of mathematical knowledge, secondly the ability to extend that knowledge independently and thirdly the ability to use that knowledge. It was also suggested that mathematics courses in tertiary education have traditionally concentrated on the first of these to the detriment of the other two. There is however, within the discipline of mathematical education today, an identifiable group of teachers and lecturers who, to a greater or lesser extent, believe that the teaching of mathematical modelling is vital to the development of an ability to use mathematics and that such teaching is necessarily a distinct activity from the teaching of other topics in mathematics. The existence of such a group, who might be characterised as the mathematical modelling movement, is a distinctively recent innovation in mathematics – it would have been difficult, for instance, to identify any such grouping prior to the mid nineteen sixties. We might ask what circumstances have led to the birth of the movement and why has its development been so rapid? In this chapter the origins of the movement will be described, the development of the theory of mathematical modelling traced and some new insights into the nature of the mathematical modelling process proposed.

2.1 The need to teach mathematical modelling

A panel discussion, organised in 1961 by the [American] Society for Industrial and Applied Mathematics, between Professors Carrier, Courant, Rosenbloom and Yang entitled 'Applied mathematics: what is needed in research and education' is reported by Greenberg (1962). Greenberg, who chaired the meeting, referred, in his introduction, to a revolution in mathematics teaching in the USA and to the need, in the light of this, to determine a design for the future of applied mathematics. It is particularly interesting to notice what skills the distinguished panellists identified as essential for applied mathematicians. In his address Prof Carrier said

To [contribute to the quantitative understanding of scientific phenomena], [applied mathematicians] must be so thoroughly informed in the fundamentals of some broad segment of the sciences . . . that they can pose the question or family of questions they pursue as a mathematical query using, as the occasion demands, either time-honoured and well established scientific laws . . . or carefully conjectured models. Such an applied mathematician must also have an understanding of mathematics, a knowledge of technique, and such skill that he can use either rigorously founded techniques or heuristically motivated methods to resolve the mathematical problem, and he must do so with a full realisation as to the implications of each with regard to reliability and interpretation of results. In particular, the applied mathematician must be very skillful at finding that question (or family of questions) such that the answer will fill the scientific need while the extraction of the answer and its interpretation are not prohibitively expensive.

Prof Carrier went on to emphasise that, in his opinion, such skills could only be inculcated in mathematics students if their study of mathematics was closely integrated with study of the sciences, engineering and other disciplines in which that mathematics found its applications.

Prof Rosenbloom voiced the opinion that even one of the weightiest and most popular of classical mathematical physics textbooks was both too much and too little for a complete professional training for a physicist. He described it as too little in the sense that

. . . there is nothing in there about how you would actually set up a problem.

He subsequently added

The basic problem is that applied mathematics is an art in which mathematics is only a part. We have a situation in the real world from which you have to create a mathematical model by idealisation and simplification. . . . We then study the mathematical model using all the power and technique of mathematics on that, often using our intuition from the interpretation that we had in mind. Then the test of our model is whether, when you interpret it back in reality, it works. And the middle part, the study of this mathematical model, which is the game of the mathematician, is only part of the whole process of applied mathematics.

In all four contributions to this discussion there is to be found an appreciation of the importance, for applied mathematics and its fields of application, both traditional and modern, of modelling and model building, and a realisation that, at the time of the discussion, something needed to be done to improve the teaching of these skills.

Ten years later Prager, in the introductory remarks to a 'Symposium on the future of applied mathematics' (Prager, 1972), described the applied mathematician thus

With the pure mathematician, the applied mathematician shares the interest in developing new mathematics, and with the scientist or engineer, the interest in applying mathematics to the improvement of our understanding and control of natural or man-made environments. As an intermediary between these groups, the applied mathematician should appreciate, though not necessarily emulate, the pure mathematician's insistence on rigour as well as the willingness of scientists and engineers to accept heuristic reasoning. He must be able to construct, not only a rigorous proof of a mathematical proposition, but also a workable mathematical model of the phenomenon he plans to investigate.

Very often, the applied mathematician's skill in the construction of suitable models will contribute as much to the success of an investigation as his knowledge of the analytical or numerical techniques that will be needed to treat the mathematical relations governing the behaviour of the adopted model. His familiarity with these techniques enables him to foresee difficulties that may result from the inclusion of certain effects in the model. He will then question his clients about the need for including these effects, warn of the mathematical consequences of this inclusion, and stress the fact that a coarse model that is readily manipulated mathematically may yield a better insight into a natural phenomenon or technical process than a more refined but mathematically unwieldy model.

Having given this description of the work of applied mathematicians, he went on to suggest how their education should be changed in order to better fit students for the coming demands.

I believe that, in teaching applied mathematics, we should devote more time to the process of discovery than is customary today. This will not be easy on account of the tradition that requires us to present our results in an orderly and logical way and makes us reluctant to report the often erratic and illogical ways in which these results were obtained. We should overcome this reluctance because an account of the process of discovery will frequently be more useful to the student of applied mathematics than the particular result.

We see that Prager, like his colleagues, appreciated the demands which the applied mathematician faced (and indeed still faces) and recognised that some changes were needed in the education of applied mathematicians to fit them better to deal with these. His prescription for these changes goes some way to meeting the demand, but the proponents of mathematical modelling would argue that it is even better for students to experience the process of mathematical discovery than to hear an account of it.

In 1970 McLone undertook a major study of the perceptions of both the employers of mathematics graduates and the graduates themselves of the relevance and usefulness of mathematics degree courses in UK universities. The study was conducted through the medium of postal surveys both of employers and of graduates from mathematics departments of UK universities in the years 1964, 1966 and 1968. The findings of the survey are reported in Griffiths and McLone (1971) and in McLone (1973). Amongst the conclusions drawn from the responses were

A transfer in emphasis to the basic problems of mathematical modelling and the formulation of mathematical problems is both necessary and desirable, if a mathematician is to be of benefit to the environment in which he works after graduation.

and

The request most often made (by employers) is for mathematics graduates with an appreciation of the applicability of their subject in other fields and an ability to express problems, initially stated in non-mathematical terms, in a form amenable to mathematical treatment with the subsequent re-expression in a readily understandable form to non-mathematical colleagues. This request is usually accompanied by a statement which indicates that this is not often found in those currently graduating from British universities.

and

. . . an important aspect of applied mathematics which is emphasised by all groups is mathematical modelling, that is, the modelling of real situations in mathematical terms. (McLone, 1973)

Writing as an industrial employer of mathematics graduates, Klamkin (1971) described what he saw as the role of the mathematician in industry. He stated

After a problem is recognised, the next stage is the formulation of something precise to work on. This consists of making a mathematical model of the physical situation which, on the one hand, has to be simple enough to permit a complete mathematical analysis but, on the other hand, is sufficiently close to reality to be relevant to the actual physical problem being considered. This model building is probably the most difficult and valuable task for the industrial mathematician.

Klamkin went on to criticise the abilities of graduates of both bachelors and doctoral degree courses in these areas.

Gaskell and Klamkin (1974) reported the findings of an informal survey carried out by the Mathematical Association of America in which the views of heads of non-academic groups of mathematicians (such groups being, in fact, primarily industrial) were sought on a number of issues related to the employment of mathematics graduates in their groups. Included in these issues were some questions relating to the adequacy and appropriateness of their education. The paper was comprised mainly of quotations from the responses received, these being collated and arranged according to issues. A number of these quotations made points relevant to a discussion of mathematical modelling. In particular the comment

Training could be improved by teaching courses in the formation of mathematical models – probably by showing students how to describe mathematically and try to optimize some of the non-mathematical activities that they are familiar with in their everyday life . . .

and a closing comment by the authors

Our traditional educational procedures have provided only training in mathematical manipulation, with extremely little attention to problem recognition, formulation, and follow-up.

are relevant to our concerns here.

Taken as a whole the preceding comments seem to indicate that there was, in the sixties and early seventies, a growing recognition that an identifiable body of skills, loosely characterised as an ability in mathematical modelling, existed and was an important acquisition for applied mathematicians whether intending to follow careers in the academic world or in industry. There was, further, an acknowledgement that traditional degree courses did not succeed nearly well enough in inculcating these skills and that some action was necessary, at tertiary and also at lower levels in the educational system, to rectify this situation. The mathematical modelling movement emerged in response to this need.

One indication of the emerging movement was the increasing number of publications, concerned with the philosophy of applied mathematics and related subjects, in which modelling was mentioned as a desirable or essential objective or component of applied mathematics education; for example, Lin (1967, 1976, 1978), Woods (1969), Ford and Hall (1970), Hall (1972) and Lighthill (1979). More recently there has been a considerable body of published literature concerned with mathematical modelling and a number of journals have been introduced covering the field (the *UMAP Journal*, the *Journal of Mathematical Modelling for Teachers* and its successor, the journal *Teaching Mathematics and Its Applications* published by the Institute of Mathematics and Its Applications, for instance). A regular series of International Conferences on the Teaching of Mathematical Modelling was started in 1983 and continues on a biennial basis.

2.2 Early expositions of the methodology of modelling

As a consequence of the increasing appreciation of the importance of mathematical modelling, initiatives have been taken to establish, in undergraduate courses, effective ways of teaching (or encouraging the learning of) the skills of modelling. A *sine qua non* of effective teaching is that teachers have an appropriate understanding of what it is that they are trying to teach. Hence part of the initiative in the teaching of mathematical modelling has been the attempt to establish a more precise understanding of what mathematical modelling is and what characterises an effective modeller. Various models of the modelling process, or methodologies of modelling, have been proposed and a line of development may be discerned.

Drawing on the work of Pollack (1959), Klamkin (1971) proposed a five stage model of the problem solving process, the stages being

1. Recognition
2. Formulation
3. Solution
4. Computation
5. Explanation

Klamkin sees the modelling, or in his terms model building, as taking place in stage two of this process though more recent expositions of modelling would probably argue that all of Klamkin's five stages are aspects of mathematical modelling. More importantly for the present discussion, however, it should be noted that Klamkin described modelling or problem solving as an essentially linear process starting at stage one and proceeding to stage five at which point