Two-Dimensional Wavelets and their Relatives

Two-dimensional wavelets offer a number of advantages over discrete wavelet transforms when processing rapidly varying functions and signals. In particular, they offer benefits for real-time applications such as medical imaging, fluid dynamics, shape recognition, image enhancement and target tracking. This book introduces the reader to 2-D wavelets via 1-D continuous wavelet transforms, and includes a long list of useful applications. The authors then describe in detail the underlying mathematics before moving on to more advanced topics such as matrix geometry of wavelet analysis, three-dimensional wavelets and wavelets on a sphere. Throughout the book, practical applications and illustrative examples are used extensively, ensuring the book's value to engineers, physicists and mathematicians alike.

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Contents

Prologue

Wavelets are everywhere nowadays. Be it in signal or image processing, in astronomy, in fluid dynamics (turbulence), in condensed matter physics, wavelets have found applications in almost every corner of physics. In addition, wavelet methods have become standard in applied mathematics, numerical analysis, approximation theory, etc. It is hardly possible to attend a conference on any of these fields without encountering several contributions dealing with them. Correspondingly, hundreds of papers appear every year and new books on the topic get published at a sustained pace, with publishers strongly competing with each other. So, why bother to publish an additional one?

The answer lies in the finer distinction between various types of wavelet transforms. There is, indeed, a crucial difference between two approaches, namely, the *continuous* wavelet transform (CWT) and the *discrete* wavelet transform (DWT). Furthermore, one has to distinguish between problems in one dimension (signal analysis) and problems in two dimensions (image processing), since the status of the literature is very different in the two cases.

Take first the one-dimensional case. Beginning with the classic textbook of Ingrid Daubechies [Dau92], several books, such as those of M. Holschneider [Hol95], B. Torrésani [Tor95] or A. Arnéodo *et al.* [Arn95], cover the continuous wavelet transform, in a more or less mathematically oriented approach. On the other hand, the discrete wavelet transform is treated in many textbooks, more in the signal processing style, such as M. V. Wickerhauser [Wic94], M. Vetterli and J. Kovačević [Vet95], P. Wojtaszczyk [Woj97], or S. G. Mallat [Mal99], whereas others emphasize the algorithmic aspects, sometimes in a rather abstract way, for example, C. K. Chui [Chu92] or Y. Meyer [Mey94] (of course, there are many more on the market). Altogether these books tell a fascinating story, that is ideally depicted in the highly popular volume of B. Burke Hubbard [Bur98], which is based on interviews by the author with all the founding "fathers" of the theory (J. Morlet, A. Grossmann, I. Daubechies, Y. Meyer, etc.).

It is a fact that DWT-inspired methods (multiresolution, lifting scheme, etc., that we shall describe in due time) constitute the overwhelming majority among the wavelet community, under the joint influence of electrical engineering (signal processing with filters and subband coding) and applied mathematics (numerical and algorithmic methods). Yet the CWT and, more generally, redundant representations of signals, offer distinct advantages in certain cases, as we shall see later.

In two dimensions, that is, application to image processing, the situation is clearer. Discrete methods are somewhat trivial, since the basic structure is that of a tensor product, 2-D $=$ 1-D $\otimes$ 1-D, enforcing a Cartesian geometry (x and y coordinates). Thus most textbooks on the DWT will cover, although briefly in general, the 2-D case as a straightforward extension of the 1-D setup. As for the 2-D CWT, it receives at best a cursory treatment in most cases. The raison d'être of the present volume is precisely to fill this gap in the literature and give a thorough treatment of the 2-D CWT and some of its applications in image processing and in various branches of physics. As a byproduct, we will also discuss in detail several extensions, such as 3-D wavelets, wavelets on the sphere or wavelets in space-time.

A historical note

Before entering the subject proper, it may not be uninteresting to give some details on its origin, without pretension to completeness, of course; we are not historians. The first extension of the wavelet transform to imaging is due to Mallat [259,260], who developed systematically a 2-D discrete (but redundant) WT, combining the traditional concept of filter bank and the analogy with human vision. In fact, most of the concepts are indeed already present in the pioneering work of Marr [Mar82] on vision modeling, in particular the idea of multiresolution. Indeed, when we look at an object, our visual system works by registering first a global, low-resolution, image and then focusing systematically to finer and finer details. Thus, contrary to the 1-D case, the 2-D discrete WT preceded the continuous version.

The 2-D continuous WT was born in a quite different way. The story starts in the coffee room of the Institut de Physique Théorique in UCL, Louvain-la-Neuve (LLN), in Spring 1987. Alex Grossmann from Marseille, one of the founding fathers of wavelets, was visiting J.-P. A., indeed they had already started to collaborate on the application of 1-D wavelets in NMR spectroscopy. Thus the two were discussing a possible Ph.D. topic for a young African student, called Romain Murenzi (R.M.). The latter had just concluded a Master's thesis on five-dimensional quantum field theory, a subject hardly practical for a developing country! So the idea came up, why not try to do in two dimensions what had been so successful in 1-D, namely, wavelet analysis? The topic seemed tractable, involving moderate amounts of mathematics and some simple computing technology, and if it worked out, there could be very interesting practical applications. The problem was that nobody knew how to do it! The next summer, R.M. went down to Marseille and started to work with Grossmann and Ingrid Daubechies who happened to be there too. And when he came back 3 months later, the solution was clear. The key is to start from the operations that one wants to apply to an image, namely, translations in the image plane, rotations for choosing a direction of sight, and global magnification (zooming in and out). The problem is to combine these three elements in such a way that the wavelet machine could start rolling (there are mathematical conditions to satisfy here). The result of R.M. was that the so-called similitude group yields a solution (actually, the only one). There remained to put it all together, to turn the mathematical crank and to apply the resulting formalism to a real problem, namely, 2-D fractals (the outcome of a visit of R.M. to Arnéodo in Bordeaux), and the Ph.D. thesis was within reach [Mur90]. Several papers followed [12,13], more M.Sc. or Ph.D. students got involved over the years. We may cite Pierre Carrette, Stéphane Maes, Canisius Cishahayo, Pierre Vandergheynst, Yébéni B. Kouagou, Laurent Jacques, Laurent Demanet. Each of them has brought his contribution to the edifice, small or big, but always useful.

This is probably a good place for asking, why wavelets? After all, there are plenty of methods available for processing images. What is new here? A key fact is probably that wavelets are somehow a byproduct of quantum thinking. More precisely, it is an application of the quantum idea of a *probe* for testing an object, the result being given by the scalar product of the two functions (indeed the framework is a Hilbert space, that of finite energy signals). To get the transform, the probe is translated and scaled (zoom), and turned around in the 2-D case, and the result is plotted as a function of the corresponding parameters. (Actually the same could be said of the so-called Gabor or Windowed Fourier transform.) One gets in this way a highly flexible and efficient tool for signal/image processing, that sheds a different light and offers an alternative approach to many standard problems, in particular those involving the detection of singularities or discontinuities in signals. As somebody once remarked, wavelets do not solve all the problems, but they often help asking the right questions.

Another sign of the quantum influence is the crucial role played by a unitary group representation, a tool largely absent in classical physics – and thus from signal processing as well. And it is no accident, in our opinion, that the crucial steps in developing wavelets were made by Alex Grossmann and Ingrid Daubechies, both educated as theoretical (quantum) physicists. Otherwise, it might have taken much longer for electrical engineers and mathematicians to meet!

About the contents of the book

Now it is time to give some indications on the contents of the book. One can divide it into several stages. In a first part (Chapters 1–3), we develop systematically the continuous wavelet transform, first in one dimension (briefly), then in two dimensions. The emphasis here is on the practical use of the tool, with a minimum of mathematics. Then we devote two long chapters, 4 and 5, to applications. Three short chapters, 6–8, set the general mathematical scene. This allows us, in Chapters 9 and 10, to describe wavelets in more general settings (3-D, sphere, space–time). In Chapter 11, finally, we discuss some recent developments that actually go beyond wavelets. This gradual structure is one of the original aspects of the book, in comparison with those on the market.

Let us go into more details. As a warming up exercise, we begin, in Chapter 1, with a rather concise overview of the 1-D WT. This allows the reader to develop a feeling about the wavelet transform and to understand its success in signal processing. All aspects will be touched upon: the continuous WT, multiresolution and the discrete WT, various generalizations of the latter, some applications. One of the leitmotives is the role of *redundancy*, especially with respect to stability of the representation.

Chapter 2, which forms the hard core of the first part, presents in a systematical way the theory of the 2-D CWT. As said above already, the starting point is to decide which elementary operations one wants to apply to an image. Choosing translations in the image plane, rotations (direction of sight), and global magnification (zooming in and out), together with the probe idea, leads uniquely to the 2-D CWT. We study in detail its basic properties: energy conservation, reconstruction formula, reproducing property, covariance under the chosen operations. Then we describe the interpretation of the WT as a singularity scanner and as a phase space representation of signals. Since the WT of a 2-D image is a function of four variables, visualizing it inevitably becomes problematic. Hence the need to reduce the number of parameters, either by fixing some of them, or integrating over them. This introduces a tool that will prove very useful in the applications, namely, the various partial energy densities, that is, the function obtained by integrating the squared modulus of the CWT over a subset of the parameters. In other words, various types of *wavelet spectra*, the analogs of the familiar power spectrum of a signal.

As is well known in 1-D, the CWT is highly redundant, as one can expect from a transform that doubles the number of variables: one to two in 1-D, two to four in 2-D. This fact may be exploited in two ways. Either one limits oneself to a small subset of the transform, where most of the energy is concentrated, and thus one is led to the notions of local maxima, ridges and skeleton; or one discretizes the CWT and obtains *wavelet frames*. Such a representation is still redundant, but much less than the full CWT, and in many instances is a good substitute for a genuine orthonormal basis. An alternative is the so-called *dyadic WT*, originally due to Mallat, in which only the scale variable is discretized. Together with the latter, we also describe briefly the standard DWT, based on the multiresolution idea, and several generalizations, mostly the so-called *lifting scheme*. We conclude the chapter with a thorough discussion of a different scheme, called directional dyadic wavelet frames. Here, as in 1-D, there are two conflicting requirements: redundancy of the transform, which brings stability, and computing economy, that seeks fast algorithms. The formalism described here offers a good compromise.

When it comes to treating a precise problem, the first question to ask is, which wavelet should one use? Thus there is a need for a sizable collection of them, well documented and calibrated. The aim of Chapter 3 is to provide this. The crucial distinction here is whether directions in the image are relevant or not. If they are not, a pointwise analysis suffices, and one can use rotation invariant (isotropic, radial) wavelets, the best known being the Mexican hat or LOG wavelet (already introduced by Marr [Mar82]). On the contrary, if directions must be detected, one needs a wavelet with a good orientation selectivity. The most efficient result is obtained with the so-called *directional wavelets*. These are filters living in a convex cone, with apex at the origin, in Fourier space. Examples are the 2-D Morlet wavelet and the family of conical wavelets. All these wavelets, and some more, are discussed in detail in Chapter 3, and their performances determined quantitatively.

At this stage, the tool is ready and we turn to applications. Many of them are not easy to find, because they have appeared only in conference proceedings or in (unpublished) Ph.D. theses. For that reason, we have decided to present them in a rather detailed fashion, always giving original references, including personal websites when available. In each case, we emphasize the rationale for using wavelets in the particular problem at hand, rather than go into the technicalities.

It is convenient (although not always unambiguous) to distinguish between two different fields of applications, image processing and physics. To the first type, the subject matter of Chapter 4, belong contour detection and character recognition; automatic target detection and recognition (for instance, in infrared radar imagery); image retrieval from data banks; medical imaging; detection of symmetries in patterns, in particular quasicrystals and other quasiperiodic patterns; and image denoising. The chapter concludes with two nonlinear extensions of the CWT, which both have important applications. The first one is *contrast enhancement* in images through an adaptive normalization. This technique, based on analogy with our visual system, may be of interest in medical imaging. Indeed typical images, such as those obtained by radiography or by NMR imaging, have rather weak contrast, which makes their interpretation sometimes difficult. The other problem we deal with is *watermarking* of images, which consists in adding an invisible "signature" (the watermark) to an image, that only the owner can recognize and is robust to manipulations. Clearly the field of image copyright offers a good market for such techniques. The novel method we present is based on the contrast analysis described previously, exploiting directional wavelets, and it turns out to be particularly efficient.

The second class of applications, described in Chapter 5, concerns various fields of physics. Characteristically, they all belong to classical physics, as opposed to quantum physics, because the former relies much more on images. Indeed, there are very few applications of wavelet analysis in quantum problems.

The first domain on which 2-D wavelets have made a substantial impact is astronomy and astrophysics, for several reasons. The Universe has a marked hierarchical structure.

Nearby stars, galaxies, quasars, galaxy clusters and superclusters have very different sizes and live at very different distances. Thus the scale variable is essential and a multiscale analysis is in order. This, of course, suggests wavelet analysis, and indeed many authors have used it in problems such as determination of the large-scale structure of the Universe, galaxy or void counting, or analysis of the cosmic background radiation. In addition, we describe more in depth two applications of our own, namely, the detection of various magnetic features of the Sun, from satellite images, and the detection of distant gamma-ray sources in the Universe. In the latter case, difficult statistics problems arise, because of the extreme weakness of the signal (such a source emits very few high energy photons).

The next topic is Earth physics: fault detection in geophysics, seismology, climatology (notably, thunderstorm prevision). A number of successful applications pertain to fluid dynamics, from the detection of coherent strucures in fully developed 2-D turbulence (a domain pioneered forcefully by Marie Farge [164]) to the measurement of the velocity field in a turbulent fluid, or the disentangling of a 2-D (or 3-D) wave train. Next comes the world of fractals. These are structures that are solely characterized by their behavior under a scaling transformation: ideal ground for wavelets! However, the self-reproducing properties of physical fractals are in general only approximate, so that methods from statistical mechanics are needed. Thus, a thermodynamical formalism has been designed by Arnéodo and his group in Bordeaux for treating such problems, and we give a brief account of it. Finally we touch upon the problem of shape recognition, where wavelet descriptors have proven useful too.

At this point, the book undergoes a sort of phase transition. Up to here, everything was done by hand, so to speak. The properties of the CWT have been derived by explicit calculations and very few mathematical prerequisites have been asked for. But now it is time to look over the hill and notice that the whole theory is firmly grounded in group theory. Indeed the wavelet transform and all its properties may be entirely derived from an appropriate representation of the affine group, both in one and in two dimensions. A mathematical condition, called square integrability of the representation, ensures the validity of the derivation, in particular the possibility of inverting the wavelet transform, that is, of obtaining reconstruction formulas. We devote two rather short chapters, 6 and 7 to these developments, with a double benefit. First, on the pedagogical level, we want to convince the reader that the group-theoretical approach is not only mathematically correct and pleasant, it is also natural and easy. It allows us indeed to understand in a simple and unified language the deeper mathematical structures involved. It is also quite efficient, in that it yields a general formalism (in fact, a special case of the coherent state formalism, well known in quantum physics, in particular, in quantum optics) that permits us to extend the CWT to more general manifolds, such as $\mathbb{R}^3$, the two-sphere, or space–time, all generalizations that will be discussed in later chapters. Of course, we do not expect our reader to be fully conversant with group theory, and we will define all the needed ingredients along the way. Actually we will essentially restrict our treatment to 2×2 or 3×3 matrices, without resort to abstract notions. Nevertheless, we found it convenient to gather all the group-theoretical information in a separate appendix.

We begin, in Chapter 6, by revisiting the 1-D CWT in the light of the so-called $ax + b$ or restricted affine group of the line, that is, the set of all translations and positive dilations. It turns out that the CWT may also be interpreted as a phase space representation of signals, in the sense of Hamiltonian mechanics, and the group-theoretical language makes this evident. The same treatment is then applied to the Gabor transform, also called Short Time or Windowed Fourier transform, simply replacing the affine group by the Weyl–Heisenberg group, that is, the group of phase space translations (this point of view has also been emphasized by Daubechies [Dau92]). Next, in Chapter 7, we repeat the procedure in two dimensions. Here the relevant group is the similitude group SIM(2), which consists of translations, rotations and dilations of the plane, that is, precisely all the transformations we have chosen to apply to images. Here, as in the 1-D case, the basic tool is a representation of the group by unitary operators acting in the space of finite energy signals, a natural representation that possesses the property of square integrability, meaning roughly that its matrix elements are square integrable functions of the group parameters. Here too, the CWT is a phase space realization of signals, and we spend some time exploring the consequences of this fact.

In a third chapter with a mathematical flavor, Chapter 8, we discuss two less known properties of wavelets. First, some of them have minimal uncertainty, in the sense that they saturate some uncertainty relations linked to the Lie algebra of the wavelet group, exactly as Gaussians saturate those associated to the canonical commutation relations. Then we explore the relationship between wavelet transforms and the Wigner transform, well–known in physics and in radar theory (under the name of the closely related ambiguity function).

The next two chapters are devoted to various extensions of the standard CWT, that can be derived with help of the general formalism just developed. First we treat, in Chapter 9, the higher dimensional cases. We begin with the 3-D CWT, which is a straightforward extension of the 2-D case. Then we examine in depth the CWT over the 2-sphere. Here, of course, there is a strong motivation from several domains, from geophysics to astrophysics. The former is clear. As for the latter, when one considers the whole Universe, as in the problem of gamma source detection mentioned above, it is necessary to take the curvature into account.

However, there is an equally appealing aspect in the mathematics of the subject. Indeed, the group to consider here is the conformal group of the sphere S^2, which is nothing but the proper Lorentz group $SO_o(3, 1)$. The same group is also the conformal group of the plane $\mathbb{R}^2$, for instance, the tangent plane at the North Pole. The sphere and its tangent plane are mapped onto each other by the stereographic projection from the South Pole and its inverse. This operation is in fact the key to the construction of a spherical CWT. Indeed, the operations to be performed on spherical signals are motions on the sphere, given by rotations, and local dilations around a given point. In order to define these, one first defines dilations around the North Pole by lifting the corresponding ones in the tangent plane by inverse stereographic projection. Then, dilations around any other point of the sphere are obtained by combining the previous ones with an appropriate rotation. As a consequence, the parameter space of the spherical CWT is not the Lorentz group itself, but a homogeneous space of it, containing only rotations and the dilations just defined, that is, the quotient of $SO_o(3, 1)$ by a certain subgroup. Therefore, one needs the general formalism described in Chapter 7 in order to get a genuine spherical CWT. As an additional benefit, one recovers the natural link between the sphere and its tangent plane: the spherical CWT tends to the usual plane CWT when the radius of the sphere increases to infinity (the so-called Euclidean limit). It is gratifying that this aspect too is entirely described by the group-theoretical machinery, in terms of an operation called group contraction. Another byproduct of our spherical CWT is the possibility of designing good wavelet approximations of integrable functions on the sphere, another result previously known in the plane case. Here again practical applications are at hand, in the context of the so-called Geomathematics advertised by Freeden and his school [Fre97].

Then we turn, in Chapter 10, to the extension of the CWT to space–time. The problem of interest here is, of course, motion estimation, more precisely, detection, tracking, and identification of objects in (relative) motion. Examples include traffic monitoring, autonomous vehicle navigation, and tracking of ballistic missile warheads. This is a difficult problem, since the data is huge and often very noisy. As a consequence, most algorithms tend to lose track of the targets after a while, particularly if the latter changes its appearance (e.g., a maneuvering aeroplane) or in the case of an occlusion (one moving object hides another one). From the wavelet point of view, one designs a spatio-temporal CWT, whose parameters are space and time translations, rotations, global space–time dilations, that catch the size of the target, and a speed tuning parameter that measures its speed. The usual formalism goes through almost verbatim and allows one to design an efficient algorithm for motion estimation. One key ingredient again is the successive use of several partial energy densities.

In the final Chapter 11, we turn to another kind of generalizations, namely, transforms specially adapted to the detection and modeling of lines and curves, called the *ridgelet* and the *curvelet* transforms. The motivation for these new transforms, and their superiority over standard wavelets, is that they take much better into account the geometry of the object to be analyzed. A curve in the plane is more 1-D than 2-D, and the conventional 2-D CWT simply ignores this fact – hence it is unnecessarily costly. Here, of course, one experiences the much bigger richness of the 2-D world, in particular, concerning singularities of functions. These transforms naturally lead to new approaches to image compression and various nonlinear approximations, that we also describe.

We conclude the chapter and the book with a topic called ‘algebraic wavelets’. These are wavelets adapted to self-similar tilings on the line or the plane obtained by replacing the usual natural numbers by a different system of numeration, for instance, the *golden mean* $\tau = \frac{1}{2}(1 + \sqrt{5})$. This is actually a generalization of the discrete WT, but it provides another example of wavelets adapted to a specific geometry, hence it is not out of place in this volume, and we found it interesting to give a short account of it, both in 1-D and in 2-D. In the latter case, typical examples are the famous Penrose tilings of the plane, with pentagonal symmetry, and this brings us back to the study of aperiodic patterns and to quasicrystals!

The conclusion of the whole story is definitely optimistic. Wavelets, and in particular the continuous WT, have proven to be a versatile and extremely efficient tool for image processing, provided one uses the right wavelet on the right problem. Their future is undoubtly bright, in many fields of science and technology.

Before concluding this introduction, several technical remarks are in order. First, most examples that are not reproduced from original papers have been computed using our own wavelet toolbox, called the `YAW (Yet Another Wavelet) Toolbox`, and freely accessible on the Louvain-la-Neuve website `<http://www.fyma.ucl.ac.be/projects/yawtb/>`.

Next, we have found it useful to split the references into two sections, devoted to books and Ph.D. theses, and regular journal articles (with a different presentation, viz. [Ald96] and [2], respectively). As we have already said, theses are an extremely rich source of information, although they are often only accessible on the web. In general, we have tried to trace most of the results to the original papers. Of course, there are omissions and misrepresentations, due to our ignorance and prejudices. We take responsibility for this and apologize in advance to those authors whose work we might have mistreated.

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