

## I

# Introduction and Overview of Turbulence

## Introduction

In this chapter we first briefly recall, in Section 1, the derivation of the Navier–Stokes equations (NSE) starting from the basic conservation principles in mechanics: conservation of mass and momentum. Section 2 contains some general remarks on turbulence, and it alludes to some developments not presented in the book. For the benefit of the mathematically oriented reader (and perhaps others), Section 3 provides a fairly detailed account of the Kolmogorov theory of turbulence, which underlies many parts of Chapters III–V. For the physics-oriented reader, Section 4 gives an intuitive introduction to the mathematical perspective and the necessary tools. A more rigorous presentation appears in the first half of Chapter II and thereafter as needed. For each of the aspects that we develop, the present chapter should prove more useful for the nonspecialist than for the specialist.

### 1 Viscous Fluids. The Navier–Stokes Equations

Fluids obey the general laws of continuum mechanics: conservation of mass, energy, and linear momentum. They can be written as mathematical equations once a representation for the state of a fluid is chosen. In the context of mathematics, there are two classical representations. One is the so-called Lagrangian representation, where the state of a fluid “particle” at a given time is described with reference to its initial position. The other representation (adopted throughout this book) is the so-called Eulerian representation, where at each time  $t$  and position  $\mathbf{x}$  in space the state – in particular, the velocity  $\mathbf{u}(\mathbf{x}, t)$  – of the fluid “particle” at that position and time is given.

In the Eulerian representation of the flow, we also represent the density  $\rho(\mathbf{x}, t)$  as a function of the position  $\mathbf{x}$  and time  $t$ . The conservation of mass is expressed by the continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0. \quad (1.1)$$

The conservation of momentum is expressed in terms of the acceleration  $\boldsymbol{\gamma}$  and the Cauchy stress tensor  $\boldsymbol{\sigma}$ :

$$\rho \gamma_i = \sum_{j=1}^3 \frac{\partial \sigma_{ij}}{\partial x_j} + f_i, \quad i = 1, 2, 3. \quad (1.2)$$

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Here  $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$  and  $\boldsymbol{\sigma} = (\sigma_{ij})_{i,j=1,2,3}$ , componentwise in the 3-dimensional case. Moreover,  $\mathbf{f} = (f_1, f_2, f_3)$  represents volume forces applied to the fluid.

The acceleration vector  $\boldsymbol{\gamma} = \boldsymbol{\gamma}(\mathbf{x}, t)$  of the fluid at position  $\mathbf{x}$  and time  $t$  can be expressed, using purely kinematic arguments, by the so-called material derivative

$$\boldsymbol{\gamma} = \frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}, \quad (1.3)$$

or, componentwise,

$$\gamma_i = \frac{\partial u_i}{\partial t} + \sum_{j=1}^3 u_j \frac{\partial u_i}{\partial x_j}, \quad i = 1, 2, 3.$$

Inserting this expression into the left-hand side (LHS) of equation (1.2) yields the term  $\rho(\mathbf{u} \cdot \nabla)\mathbf{u}$ , which is the only nonlinear term in the Navier–Stokes equations; this term is also called the *inertial term*. The Navier–Stokes equations are among the very few equations of mathematical physics for which the nonlinearity arises not from the physical attributes of the system but rather from the mathematical (kinematical) aspects of the problem.

Further transformations of the conservation of momentum equation necessitate additional physical arguments and assumptions. Rheology theory relates the stress tensor to the velocity field for different materials through the so-called stress–strain law and other constitutive equations. Assuming the fluid is Newtonian, which is the case of interest to us, amounts to assuming that the stress–strain law is linear. More precisely, for Newtonian fluids the stress tensor is expressed in terms of the velocity field by the formula

$$\sigma_{ij} = \mu \left\{ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right\} + (\lambda \operatorname{div} \mathbf{u} - p)\delta_{ij}, \quad (1.4)$$

where  $p = p(\mathbf{x}, t)$  is the pressure. Here,  $\delta_{ij}$  is the Kronecker symbol and  $\mu, \lambda$  are constants. The constant  $\mu$  is called the shear viscosity coefficient, and  $3\lambda + 2\mu$  is the dilation viscosity coefficient. For thermodynamical reasons,  $\mu > 0$  and  $3\lambda + 2\mu \geq 0$ . Inserting the stress–strain law (1.4) into the momentum equation (1.2), we obtain

$$\rho \left\{ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right\} = \mu \Delta \mathbf{u} + (\mu + \lambda)\nabla \operatorname{div} \mathbf{u} - \nabla p + \mathbf{f}. \quad (1.5)$$

Equations (1.1) and (1.5) govern the motion of compressible Newtonian fluids such as the air at high speeds (Mach number larger than 0.5). If we also assume that the fluid is incompressible and homogeneous, then the density is constant in space and time:  $\rho(\mathbf{x}, t) \equiv \rho_0$ . In this case, the continuity equation is reduced to the *divergence-free* condition:

$$\operatorname{div} \mathbf{u} = 0. \quad (1.6)$$

Because the density is constant, we may divide the momentum equation (1.5) by  $\rho$  and consider the so-called kinematic viscosity  $\nu = \mu/\rho_0$ ; we may then replace the pressure  $p$  and the volume force  $\mathbf{f}$  by the kinetic pressure  $p/\rho_0$  and the mass density of body forces  $\mathbf{f}/\rho_0$ , respectively. In doing so, and taking into consideration the

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divergence-free condition (1.6), we obtain the *Navier–Stokes equations for a viscous, incompressible, homogeneous flow*:

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{f}, \quad (1.7a)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (1.7b)$$

where, for notational simplicity, we represent the divergence of  $\mathbf{u}$  by  $\nabla \cdot \mathbf{u}$ . For all practical purposes, the density has actually been normalized to unity; even so, we may sometimes replace (1.7a) by (1.5), remembering then that  $\nabla \cdot \mathbf{u} = 0$  and  $\rho$  is constant.

For more details on the physical aspects of fluid mechanics, we refer the reader to the classical books of Batchelor [1988] and Landau and Lifshitz [1971].

It is readily accepted that the Navier–Stokes equations govern the motion of common fluids such as air or water, so we are faced with the persistent challenging question of recovering from (1.7) such complex motions as that of smoke dispersion in the air and the turbulent flow of a river around a bridge pillar.

The flow of fluids at the microscopic level is governed by phenomena in the realm of statistical mechanics of fluids. The appropriate statistics is given by the solution of the Boltzmann equation. That equation represents the evolution of the governing distribution function, which is dependent on the position and velocity of the particles colliding with one another as a result of thermal excitation at any finite temperature. The collisions are described by an integral collision operator. In general, the collision operator represents simultaneous collisions among many particles, necessitating the use of a many-particle distribution. As such, it is very complicated and essentially impossible to evaluate precisely. Only in the case of dilute gases can one limit oneself to considering the evolution of a single-particle distribution and to binary collisions, since many body collisions are highly unlikely. In this idealized situation, the collision operator can be approximated by first-order and second-order spatial derivatives. The former is the familiar pressure gradient and the latter is the Laplacian operating on the velocity, multiplied by a constant known as the viscosity. With that approximation in hand, we can take the appropriate moments of the one-particle Boltzmann equation and so derive first the conservation of mass equation and second the conservation of momentum equation that we recognize as the NSE (when the incompressibility condition is a valid assumption).

Although such a derivation has been carried out for dilute gases, a corresponding exercise for liquids remains an open problem. This is because binary collisions play a relatively minor role in liquids, which are much denser than gases and hence feature collisions between clusters of particles. However, for practical reasons and lacking a better option, we use the Navier–Stokes equations with a simple constant viscosity as a reasonable model for liquid flows.

The origin of viscosity imposes a limit on the domain of validity of the Navier–Stokes equations. Thus phenomena on a length scale comparable to or smaller than the collision mean free path in air at atmospheric pressure (say,  $10^{-3}$  cm) cannot be described by a continuum model such as the NSE. Subsequently we will learn about

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some natural lengths that characterize the length scale region in which flow energy dissipation is dominated by viscous phenomena. It will be important then to be sure that we are still in the regime characterized by a continuum model of the flow. A similar cautionary remark applies to the amplitude of fluctuations in turbulent flows: once we are in a regime in which those fluctuations are comparable with thermally (finite temperature) induced fluctuations, the model based on Navier–Stokes equations ceases to be relevant.

### *Nondimensional Form of the Navier–Stokes Equations*

It is sometimes convenient, both for physical discussions and mathematical transparency, to consider a nondimensional form of the conservation of momentum equation. For that purpose we introduce a reference length  $L_*$  and a reference time  $T_*$  for the flow, and we set

$$\mathbf{x} = L_* \mathbf{x}', \quad t = T_* t', \quad p = P_* p', \quad \mathbf{u} = U_* \mathbf{u}', \quad \mathbf{f} = \frac{L_*}{T_*^2} \mathbf{f}',$$

where  $P_* = U_*^2$  and  $U_* = L_*/T_*$  are a reference pressure and a reference velocity, respectively. By substitution into (1.7) we obtain for  $\mathbf{u}'$ ,  $p'$ ,  $\mathbf{f}'$  the same equation but with  $\nu$  replaced by  $\text{Re}^{-1}$ , where  $\text{Re}$  is a nondimensional number called the *Reynolds number*:

$$\text{Re} = \frac{L_* U_*}{\nu}. \quad (1.8)$$

The value of the Reynolds number depends on the choice of the reference length and velocity. Usually, if  $\Omega$  (the domain occupied by the fluid) is bounded then  $L_*$  can be taken as the diameter of  $\Omega$  or as some other large-scale length related to  $\Omega$ , such as the width of a channel. The choice of  $U_*$  (and hence of  $T_*$ ) depends on the type of forcing of the flow; it can be related to the forces applied at the boundary of  $\Omega$  or to a pressure gradient, for example. Various choices of  $L_*$  and  $U_*$  can be appropriate for a given flow, leading to various definitions of the Reynolds number, but turbulent flows result for all appropriate choices when  $\text{Re}$  is large. How large depends to some extent on the shape of the domain occupied by the fluid. Once the shape of the domain  $\Omega$  is fixed, however, rescalings in length ( $L_*$ ) and velocity ( $U_*$ ) and changes in viscosity ( $\nu$ ) affect the equations only through the single parameter  $\text{Re}$ .

Hence, different experiments may lead to the same nondimensional equations. For example, multiplying the velocity by 2 and dividing the diameter of the domain by 2 leaves the Reynolds number unchanged, so we can pass from one experiment to another; this is the Reynolds similarity hypothesis constantly used in mechanical engineering. At a given Reynolds number, flows remote from the boundaries of the domain  $\Omega$ , irrespective of the latter's shape, are similar owing to some universality properties of turbulent flows. Moreover, with flows around blunt bodies (say, a sphere), as the body's radius increases and the flow velocity and/or viscosity is adjusted so as to maintain the Reynolds number constant, the flow throughout the

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modified flow domain remains similar. That is what has made possible the design of aircraft by means of relatively small models tested in moderately sized wind tunnels.

In Chapter III, instead of the Reynolds number we will use another nondimensional number: the Grashof number (see Section 13 in Chapter II).

A heuristic argument illustrating the significance of the Reynolds number emerges by comparing the inertial and dissipation terms of the Navier–Stokes equations. The inertial term  $(\mathbf{u} \cdot \nabla)\mathbf{u}$  has dimension

$$\frac{U_*^2}{L_*},$$

while the dissipation term has dimension

$$\nu \frac{U_*}{L_*^2}.$$

The inertial term dominates when

$$\text{Re} = \frac{L_* U_*}{\nu} \gg 1.$$

However, a much more subtle analysis that is valid at each length scale is made for the Kolmogorov theory of turbulence.

By setting  $\text{Re} = +\infty$  (i.e.,  $\nu = 0$ ), we obtain the case of inviscid flows. In this case, the divergence-free condition is retained but the momentum equation changes, resulting in the Euler equations for inviscid perfect fluids:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f}, \quad (1.9a)$$

$$\nabla \cdot \mathbf{u} = 0. \quad (1.9b)$$

Note that some of the difficulties encountered in studying turbulent behavior, a largely inviscid regime, arise because the transition from Euler's equations to the Navier–Stokes equations necessitates a change from a first-order system to a second-order one in space ( $\nabla$  to  $\Delta$ ), which involves a singular perturbation.

## 2 Turbulence: Where the Interests of Engineers and Mathematicians Overlap

Principal substantive questions related to turbulence have been raised since the beginning of the twentieth century, and a large number of empirical and heuristical results were derived – motivated principally by engineering applications. This includes the work of Lamb [1957], mostly on addressing idealized inviscid flows; Prandtl [1904], on eddy viscosity and boundary layers; Taylor [1935, 1937], on viscous flows; and von Karman [1911, 1912], on the nature of the boundary layer.

At the same time, in mathematics there appears the pioneering work of Jean Leray [1933, 1934a,b] on the Navier–Stokes equations. Leray speculated that turbulence is

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due to the formation of point or “line vortices” on which some component of the velocity becomes infinite.<sup>1</sup> To enable dealing with such a situation, he suggested the concept of weak, nonclassical solutions to the Navier–Stokes equations (1.7), and this has become the starting point of the mathematical theory of the Navier–Stokes equations to this day. We will consider this approach in Chapter II and beyond. It is noteworthy that, more generally, Leray’s ideas serve also as the starting point for several important elements of the modern theory of partial differential equations. Even today, despite much effort, Jean Leray’s conjecture concerning the appearance of singularities in 3-dimensional turbulent flows has been neither proved nor disproved. Let us mention, however, the result of Caffarelli, Kohn, and Nirenberg [1982] (see also Scheffer [1977]), which considerably extends an earlier result of Leray: Given the possibility that the singular points are a fractal set (assuming that such a set exists), the 1-dimensional Hausdorff measure of that set in space and time is 0. Hence the occurrence of smooth line vortices is not possible, explaining our quotation marks around “line vortices.” Nevertheless, for all physical purposes this powerful mathematical result leaves room for a tremendously complex set of singularities, and so we remain far from closing the issues raised by Leray’s conjecture.

Before continuing with these historical notes, we remark in passing that engineers are not directly affected by such purely mathematical issues; rather, they want to calculate or measure certain physical quantities (forces, velocities, pressures, etc.). Here, however, beside the possible occurrence of singularities, another critical aspect of turbulence comes to mind: in a turbulent flow, many interesting quantities vary rapidly in time and cannot be readily measured. In practice, all that can be measured in laboratory experiments are averages (usually time averages). These averages are well-defined, reproducible quantities. This leads to the concept of ensemble averages underlying the conventional theory of turbulence, and to the concept of statistical solutions of the Navier–Stokes equations (1.7). It leads also to the idea of *ergodicity*, which is taken for granted by engineers. Loosely speaking, for all initial experimental conditions and for all sorts of reasonable ensemble averages, the experiments always yield the same measured results to within the accuracy of the measurements. We address here those questions of direct interest to engineers: the need for statistical solutions, the equivalence between ensemble averages and time averages (a question addressed in Chapter IV), and the so far unchallenged issue of the axiomatic nature of ergodicity.

We return to our brief overview of some highlights in the history of the studies of turbulent flows. It is impossible to explore here all the aspects of that history. Hence, with apologies to all whose important contributions are not mentioned here, we limit ourselves to those aspects of the history most relevant to the subject of this monograph.

<sup>1</sup> In fact, if such discontinuities occur then another question of physical nature needs to be raised concerning the validity of the Navier–Stokes equations themselves; indeed, at very short distances of order  $10^{-3}$  cm (the collision mean free path of the particles), the fluid equations are no longer pertinent.

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Turbulent flows have mystified people for ages, as evidenced for example by Leonardo da Vinci's sketches of the turbulent wakes downstream of some bridge columns. Beginning with careful experimental studies of flows under various experimental conditions (Reynolds [1883, 1895]) and with the subsequent formulation of the Navier–Stokes equations, turbulence became a subject of thorough scientific inquiry. For many years, two difficulties held the attention of various investigators. The first was a technical mathematical obstacle: the presence of the inertial term (a quadratic nonlinearity) precludes a straightforward use of the many available tools of perturbation methods. The structure of the equations demands that, at any given step in an approximation scheme, information from the next step is necessary. This had led to many attempts at formulating the so-called closure schemes, where at some step in the approximation sequence an assumption about the nature of the subsequent term is made, thereby terminating that sequence. Such an assumption, usually justified in terms of intuitive physical arguments, was then used to break the impasse in the approximation sequence. In principle, closure schemes by and large call for unprovable assumptions beyond those composing the basis for the Navier–Stokes equations. Some of the better-known closure schemes may be found in such texts as Tennekes and Lumley [1972], Leslie [1973], and Lesieur [1997], although further attempts (and controversies) in this area continue. As we shall find in the present work, the invention of the so-called inertial manifolds in the context of the rigorous theory of NSE (as well as of other nonlinear partial differential equations) opens the door to mathematically more soundly based schemes for computational approaches, offering an alternative to the conventional closure schemes.

The second obstacle to progress in the theory of turbulence was largely conceptual. Namely, how was it possible for a system described by perfectly deterministic equations to exhibit behavior that was undeniably statistical in nature? This aspect of turbulent flows, both from the experimental side and from the nascent theoretical side, is dealt with at length in the monumental work of Monin and Yaglom [1975]. Hopf [1952], followed by Foias and Prodi [1976] (see also Foias [1972, 1973, 1974]), studied an extension of Liouville's theorem that in principle yields the probability distribution function underlying the Navier–Stokes equation. Many of these efforts rested on the experimental and theoretical work of Taylor [1935, 1937] and von Karman and Howarth [1938], who clarified, on intuitive grounds, the nature of homogeneous isotropic turbulence. The simplifications resulting from the symmetries inherent in this idealized form of turbulence yielded the well-known von Karman–Howarth ordinary differential equation for the self-similar evolution of the two-point velocity correlation tensor. This idealization has also yielded Kolmogorov's theory for the spectrum of homogeneous isotropic turbulence in three dimensions (Kolmogorov [1941a,b]) (and later Batchelor's [1959] and Kraichnan's [1967] corresponding results for turbulence in two dimensions), a subject of the next section. All of these results were obtained without full understanding of the origin of the statistical nature of turbulence. A significant breakthrough occurred in the 1960s and 1970s with the discovery of stochastic instabilities in seemingly innocuous low-order ordinary

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differential equations (Lorenz [1963]) and in some nonlinear difference equations (Feigenbaum [1980]). Subsequent research (Foias and Prodi [1976], Vishik and Fur-sikov [1977a,b, 1978], Foias and Temam [1979]) on dynamical systems governed by nonlinear partial differential equations revealed that such dynamical systems may reside, in finite-dimensional function spaces, on compact attractors that may be characterized by chaotic behavior.

It is now appropriate to reiterate a point hinted at earlier, namely, the essential need for careful mathematical analysis when dealing with nonlinear entities such as the Navier–Stokes equations. While much of our physical intuition serves us well in the domain of linear phenomena modeled adequately by linear differential and partial differential equations, it can fail us – with potentially disastrous consequences – in nonlinear domains. A fairly instructive example, outside the realm of this book but worth mentioning here, concerns modeling sonic flow transition as a boundary value problem rather than (and more correctly) as an initial value problem (Greenberg and Trève [1960]). Although this may appear to be unnecessary pedantry, it clearly makes a lot of difference in the context of, say, nuclear reactor safety (Bilicki et al. [1987]). Unlike the case in linear systems, in nonlinear systems small causes can lead to very large effects indeed, as well as to qualitative differences. Because nonlinear equations can have multiple, qualitatively different solutions (different basins of attraction), a small change in initial conditions can sometimes lead to radically different time-asymptotic behavior. An even more dramatic, counterintuitive example is the previously mentioned possibility of chaotic behavior in what at first sight seem to be innocent deterministic systems (Lorenz [1963], Feigenbaum [1980], Smale [1967]). Here is a class of problems in which necessarily limited computer “experiments” can lead to misleading conclusions about the behavior of a system as a function of the governing parameters. Only a thorough analysis of the system can reveal its true nature. Occasionally, such an analysis will reveal, even without detailed numerical computations, an unphysical aspect of the system (e.g., infinite energy density, decreasing entropy, or other pathologies), which is a clear alert to the flawed nature of the system model.

In this work we concentrate on those aspects of turbulent fluid flows that can be represented in terms of so-called Sobolev spaces – that is, a class of functions satisfying the given boundary conditions – and the given physical constraints, such as divergence-free (incompressible) flow. The various norms (i.e., various integrals of some seemingly abstract quantities) in these function spaces are in fact readily recognized as tangible physical quantities that are more or less readily accessible to direct experimental observation. The relationships among these norms, and the rules for their manipulations, reveal some aspects of the turbulent flows that justify many ad hoc interpretations and inspire insights derived from direct observations of turbulence while also revealing some hitherto unrealized ones. As such, these mathematical endeavors can serve to enlarge our intuitive horizons beyond the limits of linear theories and models.

### 3 Elements of the Theories of Turbulence of Kolmogorov and Kraichnan

Turbulent flows seem to display self-similar statistical properties at length scales smaller than the scales at which energy is delivered to the flow. Kolmogorov [1941a,b] argued that, at these scales, in three dimensions, the fluids display universal statistical features. Turbulent flow is conventionally visualized as a cascade of large eddies (large-scale components of the flow) breaking up successively into ever smaller sized eddies (fine-scale components of the flow; Onsager [1945]). Such a cascade, or flow of kinetic energy from large to small scales, is taken to occur in a regime at lengths sufficiently large for the effects of viscosity to be inconsequential. The apparent energy dissipation – that is, the removal of energy from one length scale to a smaller one – is solely due to the presence of the nonlinear (inertial) term in the Navier–Stokes equations. The energy dissipation rate  $\epsilon = \nu \kappa_0^3 |\nabla \mathbf{u}(\mathbf{x}, t)|^2$  is assumed to be constant in space and time. A further essential assumption is that the cascade proceeds so that, at every length scale (or at every corresponding wavenumber), there is an equilibrium between energy flowing in from above to a given scale and that flowing out to a lower scale. Such a picture and the associated assumptions imply that, in this range of length scales (or this range of wavenumbers), the energy density at a given wavenumber can depend only on the energy dissipation rate  $\epsilon$  and the wavenumber  $k$  itself. Then dimensional analysis alone yields  $\mathcal{S}(\kappa) = \text{const.} \times \epsilon^{2/3} / \kappa^{5/3}$  for the energy density. Such a cascade process cannot continue to arbitrarily small length scales because, as the norm of the Laplacian operator increases with the decreasing length scale, eventually the effects of molecular dissipation begin to dominate the nonlinear inertial term. That length, denoted by  $\ell_d$ , is the endpoint of the inertial range and the beginning of the dissipation range.

Let us determine  $\ell_d$ . At each scale  $\ell$  (or wavenumber  $\kappa = \ell^{-1}$ ), we can define by dimensional analysis, through  $\epsilon$  and  $\ell$ , a natural time scale  $\tau$  and speed  $u$ . Indeed,  $\epsilon = \ell^2 / \tau^3$  gives  $\tau = (\ell^2 / \epsilon)^{1/3}$  and  $u = \ell / \tau = (\ell \epsilon)^{1/3}$ . Now, the dissipation length  $\ell_d$  is where the viscous term  $\nu \Delta \mathbf{u}$  starts to dominate, on average, the inertial term. Hence,

$$\nu \Delta \mathbf{u} \sim \frac{\nu u}{\ell^2} \sim \frac{\nu}{\ell \tau} > (\mathbf{u} \cdot \nabla) \mathbf{u} \sim \frac{u^2}{\ell} \sim \frac{\ell}{\tau^2}.$$

Therefore,

$$\ell^2 < \nu \tau = \nu \left( \frac{\ell^2}{\epsilon} \right)^{1/3} \iff \ell^{4/3} < \left( \frac{\nu^3}{\epsilon} \right)^{1/3}$$

and

$$\ell_d = \left( \frac{\nu^3}{\epsilon} \right)^{1/4}. \tag{3.1}$$

Kolmogorov thus inferred that, in 3-dimensional turbulent flows, the eddies of length size sensibly smaller than  $\ell_d$  are of no dynamical consequence. As we said,

the length  $\ell_d$  as defined by (3.1) is known as the Kolmogorov *dissipation length*. The corresponding wavenumber,

$$\kappa_d = \frac{1}{\ell_d} = \left( \frac{\epsilon}{\nu^3} \right)^{1/4}, \tag{3.2}$$

is the Kolmogorov *dissipation wavenumber*.

The inertial range, within which inertial effects dominate, is the range  $\ell_1 < \ell < \ell_d$ , where  $\ell_1 = L_1$  is the wavelength at which energy is injected in the flow. To each length  $\ell$  in this range we can associate a Reynolds number  $\text{Re}_\ell = u\ell/\nu$ ; hence,

$$\text{Re}_\ell^{3/4} = \ell \left( \frac{\epsilon}{\nu^3} \right)^{1/4}.$$

The largest of these Reynolds numbers obtained for  $\ell =$  the Kolmogorov macro-scale length  $L_* \simeq L_1$  is the Reynolds number  $\text{Re}$  of the flow. Hence, with (3.1),

$$\text{Re} = \left( \frac{L_*}{\ell_d} \right)^{4/3}, \quad \text{or} \quad L_* = \text{Re}^{3/4} \ell_d. \tag{3.3}$$

This relationship leads naturally to the heuristic estimate of the number of degrees of freedom in 3-dimensional flows, which is  $\text{Re}^{9/4}$ . As we shall see, this heuristic estimate is actually an upper bound on the sufficient (but not necessary) number of degrees of freedom in 3-dimensional turbulent flows.

We now present a somewhat more elaborate derivation (but one that is still divorced from the Navier–Stokes equations) of the so-called Kolmogorov spectrum.

Let  $\epsilon$  denote the average of the energy per unit mass. Then, according to the Kolmogorov theory, the length  $\ell_d$  at which the turbulent eddies are rapidly annihilated by the viscosity should be a universal function of  $\epsilon$  and the kinematic viscosity  $\nu$ , namely:

$$\ell_d = f(\nu, \epsilon). \tag{3.4}$$

In particular,  $f$  should be independent of the choice of units for space and time. Thus, if we pass from  $\mathbf{x}, t$  to  $\mathbf{x}' = \xi \mathbf{x}$  and  $t' = \tau t$  then we should still have

$$\ell'_d = f(\nu', \epsilon'). \tag{3.5}$$

Here  $\nu'$  and  $\epsilon'$  are not independent of  $\nu$  and  $\epsilon$ , and dimensional analysis yields

$$\ell'_d = \xi \ell_d, \quad \nu' = \frac{\xi^2}{\tau} \nu, \quad \epsilon' = \frac{\xi^2}{\tau^3} \epsilon; \tag{3.6}$$

that is,

$$\xi f(\nu, \epsilon) = f(\xi^2 \tau^{-1} \nu, \xi^2 \tau^{-3} \epsilon). \tag{3.7}$$

With the choices

$$\frac{\xi^2}{\tau} = \frac{1}{\nu} \quad \text{and} \quad \frac{\xi^2}{\tau^3} = \frac{1}{\epsilon}, \quad (\text{i.e., } \tau = (\epsilon \nu)^{1/2} \text{ and } \xi = \epsilon^{1/4} / \nu^{3/4}),$$