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Introduction

From the very beginning classical nonlinear dynamics has enjoyed much popularity even among the noneducated public as is documented by numerous articles in well-renowned magazines, including nonscientific papers. For its nonclassical counterpart, the quantum mechanics of chaotic systems, termed in short ‘quantum chaos’, the situation is completely different. It has always been considered as a more or less mysterious topic, reserved to a small exclusive circle of theoreticians. Whereas the applicability of classical nonlinear dynamics to daily life is comprehensible for a complete outsider, quantum chaos, on the other hand, seems to be of no practical relevance at all. Moreover, in classical nonlinear dynamics the theory is supported by numerous experiments, mainly in hydrodynamics and laser physics, whereas quantum chaos at first sight seems to be the exclusive domain of theoreticians. In the beginning the only experimental contributions came from nuclear physics [Por65]. This preponderance of theory seems to have suppressed any experimental effort for nearly two decades. The situation gradually changed in the middle of the eighties, since when numerous experiments have been performed. An introductory presentation also suited to the experimentalist with no or only little basic knowledge is still missing.

It is the intention of this monograph to demonstrate that there is no reason to be afraid of quantum chaos. The underlying ideas are very simple. It is essentially the mathematical apparatus that makes things difficult and often tends to obscure the physical background. Therefore the philosophy adopted in this presentation is to illustrate theory by experimental results whenever possible, which leads to a strong accentuation of billiard systems for which a large number of experiments now exist. Consequently, results on microwave billiards obtained by the author’s own group will be frequently represented. This should not be misunderstood as an unappropriate preference given to their own work. The billiard, though being conceptually simple, nevertheless ex-

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hibits the full complexity of nonlinear dynamics, including its quantum mechanical aspects. Probably there is no essential aspect of quantum chaos which cannot be found in chaotic billiards.

The nonexpert for whom this book is mainly written may ask whether quantum chaos is really an interesting topic in its own right. After all, quantum mechanics has now existed for more than 60 years and has probably become the best tested physical theory ever conceived. Quantum mechanics can handle not only the hydrogen atom which is classically integrable but also the classically nonintegrable helium atom. We may even ask whether there is anything like quantum chaos at all. The Schrödinger equation is a linear equation leaving no room for chaos. The correspondence principle, on the other hand, demands that in the semiclassical region, i.e. at length scales large compared to the de Broglie wavelength, quantum mechanics continuously develops into classical mechanics.

That is why there has even been a debate whether the term 'quantum chaos' should be used at all. In 1989 the leading scientists in the field came together to discuss these questions at a summer school in Les Houches [Gia89]. The proceedings are titled 'chaos and quantum physics' thus avoiding the dubious term. Berry [Ber89] once again proposed the term 'quantum chaology', introduced by him previously [Ber87]. This would obviously have been a much better choice than 'quantum chaos', but was not generally accepted. In the following years the debate ceased. Today the term 'quantum chaos' is generally understood to comprise all problems concerning the quantum mechanical behaviour of classically chaotic systems. This view will also be adopted in this book. For billiard experiments another aspect has to be considered. Most of them are analogue experiments using the equivalence of the Helmholtz equation with the stationary Schrödinger equation. That is why the term 'wave chaos' is sometimes preferred in this context. Most of the phenomena discussed in this book indeed apply to all waves and are not primarily of quantum mechanical origin.

The problems with the proper definition of the term 'quantum chaos' have their origin in the concept of the trajectory, which completely loses its significance in quantum mechanics. Only in the semiclassical region do the trajectories eventually reappear, an aspect of immense significance in the context of semiclassical theories. For purposes of illustration, let us consider the evolution of a classical system with N dynamical variables $x_1, \ldots, x_N$ under the influence of an interaction. Typically the x_n comprise all components of the positions and the momenta of the particles. Consequently the number of dynamical variables is $N = 6M$ for a three-dimensional M particle system.

Let $\boldsymbol{x}(0) = [x_1(0), \ldots, x_N(0)]$ be the vector of the dynamical variables at the

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time $t = 0$. At any later time t we may write $\boldsymbol{x}(t)$ as a function of the initial conditions and the time as

$$\boldsymbol{x}(t) = \boldsymbol{F}[\boldsymbol{x}(0),\ t]. \tag{1.1}$$

If the initial conditions are infinitesimally changed to

$$\boldsymbol{x}_1(0) = \boldsymbol{x}(0) + \boldsymbol{\xi}(0), \tag{1.2}$$

then at a later time t the dynamical variables develop according to

$$\boldsymbol{x}_1(t) = \boldsymbol{F}[\boldsymbol{x}(0) + \boldsymbol{\xi}(0),\ t]. \tag{1.3}$$

The distance $\boldsymbol{\xi}(t) = \boldsymbol{x}_n(t) - \boldsymbol{x}(t)$ between the two trajectories is obtained from Eqs. (1.1) and (1.3) in linear approximation as

$$\boldsymbol{\xi}(t) = (\boldsymbol{\xi}(0)\boldsymbol{\nabla})\boldsymbol{F}[\boldsymbol{x}(0),\ t], \tag{1.4}$$

where $\boldsymbol{\nabla}$ is the gradient of $\boldsymbol{F}$ with respect to the initial values. Written in components Eq. (1.4) reads

$$\xi_n(t) = \sum_m \frac{\partial F_n}{\partial x_m} \xi_m(0). \tag{1.5}$$

The eigenvalues of the matrix $M = (\partial F_n/\partial x_m)$ determine the stability properties of the trajectory. If the moduli of all eigenvalues are smaller than one, the trajectory is stable, and all deviations from the initial trajectory will rapidly approach zero. If the modulus of at least one eigenvalue is larger than one, both trajectories will exponentially depart from each other even for infinitesimally small initial deviations $\boldsymbol{\xi}(0)$. Details can be found in every textbook on nonlinear dynamics (see Refs. [Sch84, Ott93]).

In quantum mechanics this definition of chaos becomes obsolete, since the uncertainty relation

$$\Delta x \Delta p \geqslant \tfrac{1}{2}\hbar \tag{1.6}$$

prevents a precise determination of the initial conditions. This can best be illustrated for the propagation of a point-like particle in a box with infinitely high walls. For obvious reasons these systems are called billiards. They will accompany us throughout this book. For the quantum mechanical treatment two steps are necessary. First the Schrödinger equation

$$-\frac{\hbar^2}{2m}\Delta\psi = \imath\hbar\frac{\partial\psi}{\partial t} \tag{1.7}$$

has to be solved with the Dirichlet boundary condition

$$\psi|_S = 0, \tag{1.8}$$

where S denotes the walls of the box. Stationary solutions of the Schrödinger equation are obtained by separating the time dependence,

$$\psi_n(x,\ t) = \psi_n(x)e^{\imath\omega_n t}. \tag{1.9}$$

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Insertion into Eq. (1.7) yields

$$(\Delta + k_n^2)\psi_n(x) = 0 \tag{1.10}$$

where ω_n and k_n are connected via the dispersion relation

$$\omega_n = \frac{\hbar}{2m} k_n^2. \tag{1.11}$$

Equation (1.10) is also obtained if we start with the wave equation

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi = 0, \tag{1.12}$$

where c is the wave velocity, and if we separate again the time dependence by means of the ansatz (1.9). In contrast to the quadratic dispersion relation (1.11) for the quantum mechanical case we now have the linear relation

$$\omega_n = ck_n \tag{1.13}$$

between ω_n and k_n. It is exactly this correspondence between the stationary Schrödinger equation and the stationary wave equation, also called the Helmholtz equation, which has been used in many billiard experiments to study quantum chaotic problems using wave analogue systems (see Chapter 2).

As soon as the stationary solutions of the Schrödinger equation are known, a wave packet can be constructed by a superposition of eigenfunctions,

$$\psi(x, t) = \sum_n a_n \psi_n(x) e^{-\imath\omega_n t}. \tag{1.14}$$

For a Gaussian shaped packet centred at a wave number $\overline{k}$ and of width Δk the coefficients a_n are given by

$$a_n = a \exp\left[-\frac{1}{2}\left(\frac{k_n - \overline{k}}{\Delta k}\right)^2\right], \tag{1.15}$$

where a is chosen in such a way that the total probability of finding the particle in the packet is normalized to one. If the a_n are known at time $t = 0$, e.g. by a measurement of the momentum with an uncertainty of $\Delta p = \hbar\Delta k$, the quantum mechanical evolution of the packet can be calculated for any later time with arbitrary precision. Moreover, to construct wave packets with a given width, the sum in Eq. (1.14) can be restricted to a finite number of terms. Apart from untypical exceptions, the resulting function is not periodic, since in general the ω_n are not commensurable, but *quasi-periodic*. Thus the wave packet will always reconstruct itself, possibly after a long period of time. The exponential departure of neighbouring trajectories known from classical nonlinear dynamics has completely disappeared.

The wave properties of matter do not provoke an additional spreading of the probability density as we might intuitively think. On the contrary, in systems

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where the classical probability density continuously diffuses with time, e.g. by a random walk process, quantum mechanics tends to freeze the diffusion and to localize the wave packet [Cas79]. This has been established in numerous calculations and has even been demonstrated experimentally [Gal88, Bay89, Moo94]. The phenomenon of quantum mechanical localization will be discussed in detail in Chapter 4.

To demonstrate how the wave packet just constructed evolves with time, we now take the simplest of all possible billiards, a particle in a one-dimensional box with infinitely high walls. Taking the walls at the positions $x = 0$ and $x = l$, the eigenfunctions of the system are given by

$$\psi_n(x) = \sqrt{\frac{2}{l}} \sin k_n x, \qquad n = 1, 2, 3, \ldots \tag{1.16}$$

with the wave numbers

$$k_n = \frac{\pi n}{l}. \tag{1.17}$$

Insertion into Eq. (1.14) yields

$$\psi(x, t) = a\sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \exp\left[-\frac{1}{2}\left(\frac{k_n - \overline{k}}{\Delta k}\right)^2\right] \sin k_n x \, e^{-\imath\omega_n t}. \tag{1.18}$$

This equation holds for the propagation of both particle packets and ordinary waves, provided that the respective dispersion relations (1.11) or (1.13) are obeyed. The calculation is somewhat easier for ordinary waves. Therefore this situation will now be considered by putting $\omega_n = ck_n$. For particle waves the calculation follows exactly the same scheme. To simplify the calculation it will be further assumed that the average momentum is large compared to the width of the distribution, i.e. $\overline{k} \gg \Delta k$. Then the sum can be extended from $-\infty$ to $+\infty$, and we can apply the *Poisson sum relation*

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} g(n), \tag{1.19}$$

where

$$g(n) = \int_{-\infty}^{\infty} f(n) e^{2\pi\imath nm} \, dn \tag{1.20}$$

is the Fourier transform of $f(n)$. Application to Eq. (1.18) yields

$$\psi(x, t) = a\sqrt{\frac{2}{l}} \sum_{m=-\infty}^{\infty} \frac{l}{\pi} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(\frac{k - \overline{k}}{\Delta k}\right)^2\right] \sin kx \, e^{\imath(2lm - ct)k} \, dk, \tag{1.21}$$

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where the integration variable n has been substituted by $k = n\pi/l$. The integration is easily carried out using the well-known relation

$$\int_{-\infty}^{\infty} \exp[-(ak^2 + 2bk + c)]\, dk = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{a} - c\right) \tag{1.22}$$

for Gaussian integrals which also holds for complex values of a, b, c provided that $\mathrm{Re}(a) > 0$. The result is

$$\psi(x, t) = \sum_{m=-\infty}^{\infty} [\phi(x - ct_m) - \phi(l - x - ct_{m+\frac{1}{2}})], \tag{1.23}$$

where

$$t_m = t - m\frac{2l}{c}, \tag{1.24}$$

and

$$\phi(x) = 2a\sqrt{\frac{l}{\pi}}\Delta k \exp\left[\imath\bar{k}x - \frac{1}{2}(x\Delta k)^2\right]. \tag{1.25}$$

Equation (1.23) allows a straightforward interpretation. It describes the propagation of a Gaussian pulse with width $\Delta x = 1/\Delta k$ and velocity c, passing to and fro between the two walls and changing sign upon every reflection. For the propagation of particle waves the situation is qualitatively similar, but now the quadratic dispersion relation (1.11) leads to a spreading of the pulse with time and a pulse width $\Delta x(t)$ given by

$$\Delta x(t) = \frac{1}{\Delta k}\left[1 + \left(\frac{\hbar(\Delta k)^2 t}{m}\right)^2\right]^{1/4}. \tag{1.26}$$

For time $t = 0$ we obtain $\Delta x \Delta k = 1$ as for ordinary waves. This is just the quantum mechanical uncertainty relation.

By means of the Poisson sum relation two different expressions for $\psi(x, t)$ have been obtained. First, in Eq. (1.18), it is expressed in terms of a sum over the eigenfunctions of the systems, second, in Eq. (1.23), it is written as a pulse propagating with the velocity c being periodically reflected at the walls. This reciprocity, with the quantum mechanical spectrum on the one side and the classical trajectories on the other, will become one of the main ingredients of the semiclassical theory, in particular of the Gutzwiller trace formula. In the special example presented here the applied procedure worked especially well, since the set $\{k_n\}$ of eigenvalues was equidistant, leading to a perfect pulse reconstruction after every reflection. In the general case the pulse will be destroyed after a small number of reflections, but pulse reconstructions are still

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possible. The correspondence between classical and quantum mechanics will be demonstrated by two examples.

Figure 1.1 shows the propagation of a microwave pulse in a cavity in the shape of a quarter stadium [Ste95]. The measuring technique will be described in detail in Section 2.2.1. A circular wave is emitted from an antenna, propagates through the billiard, and is eventually reflected by the walls, thereby undergoing a change of sign (this can be seen especially well in Fig. 1.1(d) for the reflection of the pulse at the top and the bottom walls). After a number of additional reflections the pulse amplitude is distributed more or less equally

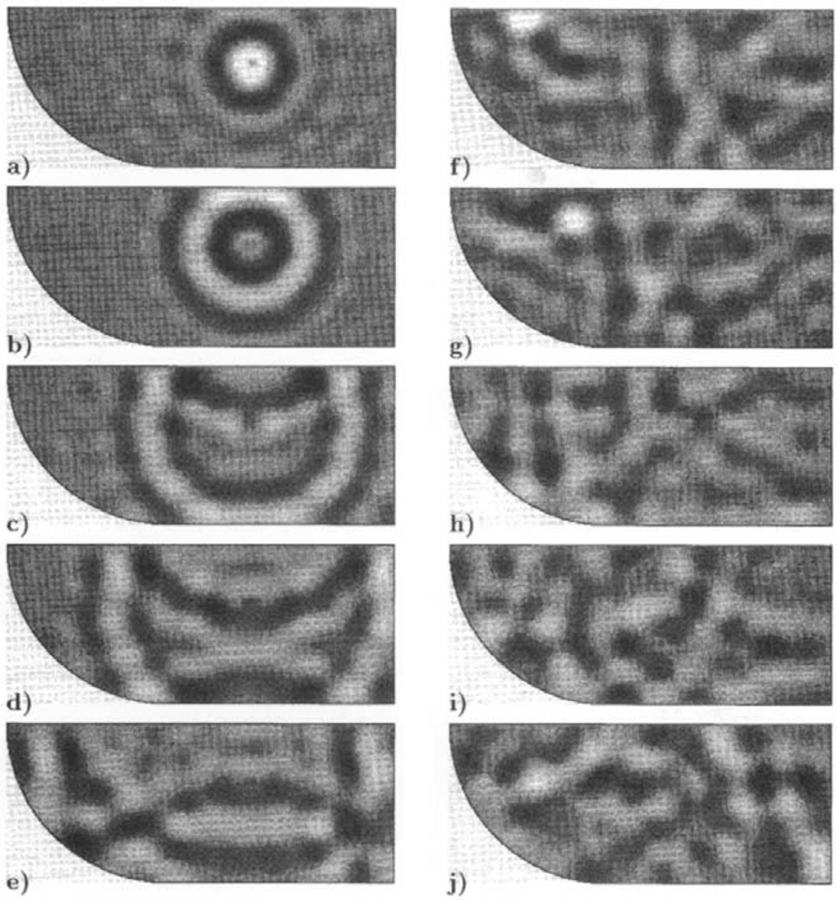

Figure 1.1. Propagation of a microwave pulse in a microwave cavity in the shape of a quarter stadium (length of the straight part $l = 18$ cm, radius $r = 13.5$ cm, height $h = 0.8$ cm) for different times $t/10^{-10}$ s: 0.36 (a), 1.60 (b), 2.90 (c), 3.80 (d), 5.63 (e), 9.01 (f), 10.21 (g), 12.0 (h), 14.18 (i), 19.09 (j) [Ste95] (Copyright 1995 by the American Physical Society).

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over the billiard. But after some time the pulse suddenly reappears (see Fig. 1.1(f)). This is even more evident in Fig. 1.2 where the pulses are shown in a three-dimensional representation for two snapshots corresponding to Figs. 1.1(a) and (f). This reconstruction has nothing to do with the quantum

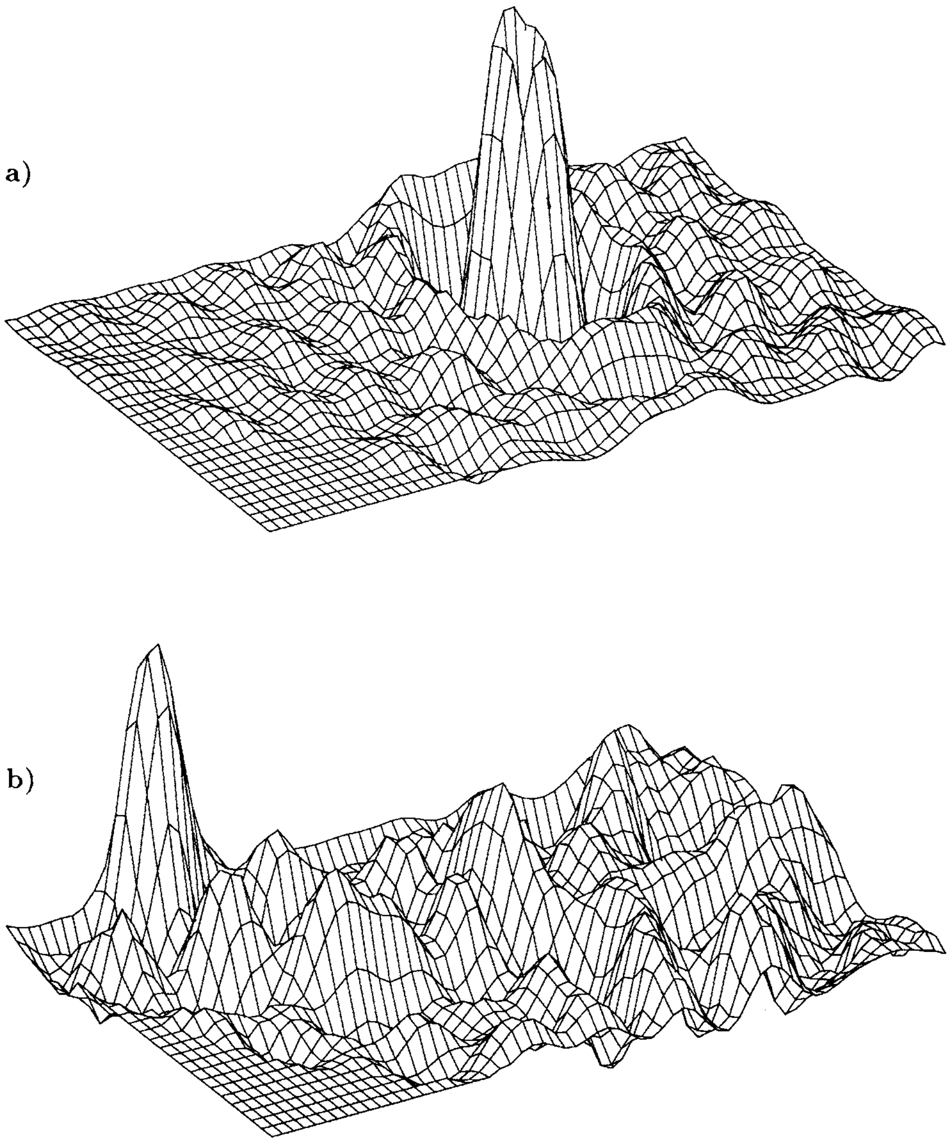

Figure 1.2. Three-dimensional view of the pulse propagation shown in Fig. 1.1 for two times, corresponding to Figs. 1.1(a) and (f), respectively [Ste95] (Copyright 1995 by the American Physical Society).

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mechanical revival discussed above, it is just a manifestation of the focussing properties of the circular boundary. All classical paths reflected by these parts of the stadium will be simultaneously focussed later in the image point of the antenna position after reflection at the circular mirror. This pulse reconstruction again demonstrates the fact that wave properties and classical trajectories are just two sides of the same coin.

The second example is taken from the kicked rotator. It is one of the most thoroughly studied chaotic systems, both classically and quantum mechanically, and will be discussed in detail in Section 4.2.1. Its Hamiltonian is given by

$$\mathscr{H}(t) = \frac{L^2}{2I} + k\cos\theta \sum_n \delta(t - nT). \tag{1.27}$$

The first term describes the rotation of a pendulum with angular momentum L, and a moment of inertia I. The second term describes periodic kicks with a period T by a pulsed gravitational potential of strength $k = mgh$. (This example convincingly demonstrates the superiority of theory over experiment. It is not so easy to realize such a situation experimentally.) The kicked rotator belongs to the Hamiltonian systems. The equations of motion are obtained from the canonical equations

$$\dot\theta = \frac{\partial \mathscr{H}}{\partial L}, \qquad \dot L = -\frac{\partial \mathscr{H}}{\partial \theta}, \tag{1.28}$$

whence follows, with the Hamiltonian (1.27)

$$\dot\theta = L, \qquad \dot L = k\sin\theta \sum_n \delta(t - n), \tag{1.29}$$

where now I and T have been normalized to one (readers not too familiar with classical mechanics will find a short recapitulation in Section 7.2.2).

As L changes discontinuously, the equations of motion define a map for the dynamical variables θ, L. Let θ_n and L_n be the values of the variables just before the $(n+1)$th kick. Immediately after the kick L_n takes the value $L_n + k\sin\theta_n$ whereas θ_n is not changed. Between the kicks L remains constant, and θ increases linearly with t. Just before the next kick the dynamical variables take the values

$$\theta_{n+1} = \theta_n + L_{n+1} \tag{1.30}$$

$$L_{n+1} = L_n + k\sin\theta_n. \tag{1.31}$$

This is Chirikov's *standard map* [Chi79]. It is especially well suited to studying the transition of a Hamiltonian system from regular to chaotic behaviour when changing one control parameter. Figure 1.3 shows Poincaré sections of the map

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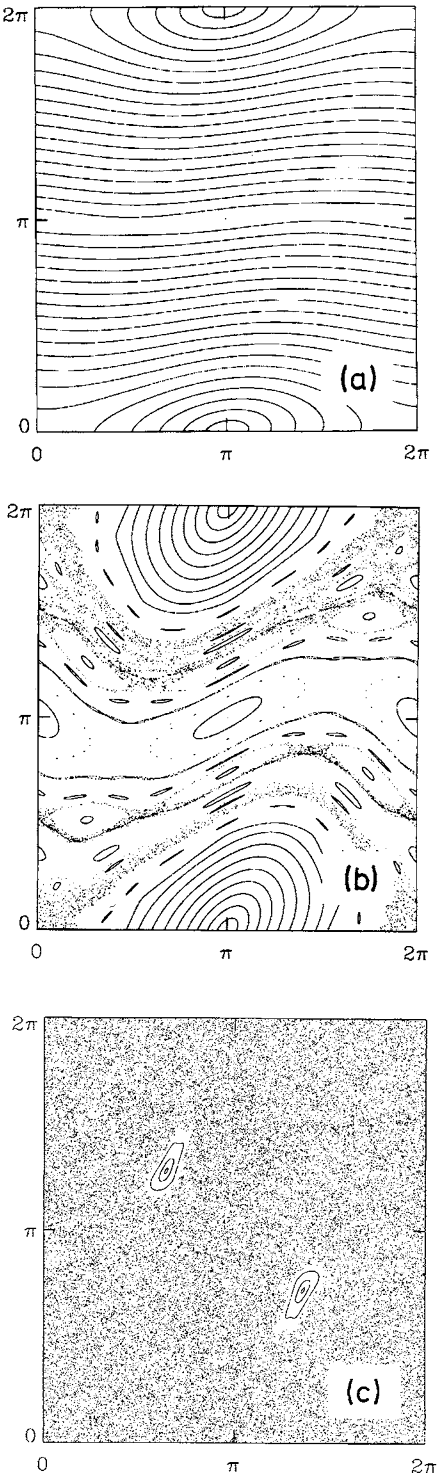

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