1

Introduction

1.1 General principles of the statistical mechanics

In the most simple terms the basic statements of the statistical mechanics can be introduced in the following way. Let the microscopic state of a macroscopic system having many degrees of freedom be described by the configurations of \( N \) variables \( \{s_i\} \), \( (i = 1, 2, \ldots, N) \). The basic quantity characterizing the microscopic states is called the energy, \( H \), and it is defined as a function of all the microscopic variables \( \{s_i\} \):

\[
H = H(s_1, s_2, \ldots, s_N) \equiv H[s]
\]

The microscopic dynamic behavior of the system is defined by some dynamic differential equations such that, in general, the energy of the system tends to a minimum. Besides, it is assumed that no observable system can be perfectly isolated from the surrounding world, and the effect of the interaction with the surroundings (the thermal bath) is believed to produce the so-called thermal noise in the exact dynamical equations. The thermal (white) noise acts as random and uncorrelated fluctuations, which produce the randomization and the mixing of the exact dynamical trajectories of the system.

Let \( A[s] \) be some observable quantity. The quantities, which are of interest in statistical mechanics, are the averaged values of the observables. In other words, instead of studying the exact evolution in time of the value \( A[s(t)] \), one introduces the averaged quantity:

\[
\langle A \rangle = \lim_{t \to \infty} \frac{1}{t} \int_0^t dt' A[s(t')]
\]  \hspace{1cm} (1.1)

which could be formally obtained after the observation during an infinite time period.

The fundamental hypothesis of the equilibrium statistical mechanics lies
Introduction

in the following. It is believed that, owing to the mixing of the dynamic trajectories, after an infinitely long observation time, the system in general ‘visits’ its different microscopic states many times, and therefore the averaged quantity in Eq. (1.1) could be obtained by averaging over the ensemble of the states instead of that over the time:

$$\langle A \rangle = \int ds_1 ds_2 \cdots ds_N A[s] P(s_1, s_2, \ldots, s_N)$$  \hspace{1cm} (1.2)

Here $P[s]$ is the probability distribution function of the microscopic states of the system. In other words, it is believed that because of the mixing of the dynamic trajectories, instead of solving the exact dynamics, the system could be statistically described in terms of the probabilities of its microscopic states given by the function $P[s]$. The probability distribution function, whatever it is, must be normalized:

$$\int ds_1 ds_2 \cdots ds_N P(s_1, s_2, \ldots, s_N) = 1$$  \hspace{1cm} (1.3)

The fundamental quantity of the statistical mechanics that characterizes the probability distribution itself is called the entropy. It is defined as the average of the logarithm of the distribution function:

$$S = -\langle \log(P[s]) \rangle \equiv -\int ds_1 ds_2 \cdots ds_N P[s] \log(P[s])$$  \hspace{1cm} (1.4)

In general, the value of the entropy could tell to what extent the state of the system is ‘ordered’. Consider a simple illustrative example. Let the (discrete) microscopic states of the system be labeled by an index $\alpha$, and let us assume that the probability distribution is such that only $L$ (among all) states have non-zero and equal probability. Then, owing to the normalization (1.3), the probability of any of these $L$ states must be equal to $1/L$. According to the definition of the entropy, one gets:

$$S = -\sum_\alpha \log P_\alpha = \log L$$

Therefore, the broader the distribution (i.e. the larger $L$) is, the larger the value of the entropy is. On the other hand, the more concentrated the distribution function is, the smaller the value of the entropy is. In the extreme case, when there is only one microscopic state occupied by the system, the entropy is equal to zero. In general, the value of $\exp(S)$ could be interpreted as the averaged number of the states occupied by the system with finite probability.

Now let us consider what the general form of the probability distribution
1.1 General principles of the statistical mechanics

function must be. According to the basic hypothesis, the average value of the energy of the system is:

\[ E \equiv \langle H \rangle = \sum_\alpha P_\alpha H_\alpha \]  

(1.5)

The interaction of the system with the thermal bath produces the following fundamental effects. First, the averaged value of its energy in the thermal equilibrium is conserved. Second, for some reason Nature is constructed in such a way that irrespective of the internal structure of the system, the value of the entropy in the equilibrium state tries to attain a maximum (bounded by the condition that the average energy is constant). In a sense, it is natural: random noise makes the system as disordered as possible. Let us now consider the form of the probability distribution function, which would maximize the entropy. To take into account the two constraints—the conservation of the average energy, Eq. (1.5), and the normalization \( \sum_\alpha P_\alpha = 1 \)—one can use the method of the Lagrange multipliers. Therefore, the following expression must be maximized with respect to all possible distributions \( P_\alpha \):

\[ S_{\beta,\gamma}[P] = -\sum_\alpha P_\alpha \log(P_\alpha) - \beta \left( \sum_\alpha P_\alpha H_\alpha - E \right) - \gamma \left( \sum_\alpha P_\alpha - 1 \right) \]  

(1.6)

where \( \beta \) and \( \gamma \) are the Lagrange multipliers. Variation with respect to \( P_\alpha \) gives:

\[ P_\alpha = \frac{1}{Z} \exp(-\beta H_\alpha) \]  

(1.7)

where

\[ Z = \sum_\alpha \exp(-\beta H_\alpha) = \exp(\gamma + 1) \]  

(1.8)

is called the partition function, and the parameter \( \beta \), which is called the inverse temperature, is defined by the condition:

\[ \frac{1}{Z} \sum_\alpha H_\alpha \exp(-\beta H_\alpha) = E \]  

(1.9)

In practice, however, it is the temperature that is usually taken as an independent parameter, whereas the average energy is obtained as the function of the temperature by Eq. (1.9).

The other fundamental quantity of the statistical mechanics is the free energy defined as follows:

\[ F = E - TS \]  

(1.10)
4

Introduction

where $T = 1/\beta$ is the temperature. Using Eq.(1.7), one can easily derive the following basic relations among the free energy, the partition function, the entropy and the average energy:

$$ F = -T \log(Z) $$

$$ S = \beta^2 \frac{\partial F}{\partial \beta} $$

$$ E = -\frac{\partial}{\partial \beta} \log(Z) = F + \beta \frac{\partial F}{\partial \beta} $$

Note, that according to the definition given by Eq. (1.10), the principle of the maximum of entropy is equivalent to that of the minimum of the free energy. One can easily confirm that taking the free energy (instead of the entropy) as the fundamental quantity which must be minimal with respect to all possible distribution functions, the same form of the probability distribution as given by Eq. (1.7) is obtained.

1.2 The mean-field approximation

In magnetic materials the microscopic state of the system is supposed to be defined by the values of the local spin magnetizations. In many magnetic materials the electrons responsible for the magnetic behavior are localized near the atoms of the crystal lattice, and the force that tends to orient the spins is the (short range) exchange interaction.

The most popular classical models, which describe this situation qualitatively, are called the Ising models. The microscopic variables in these systems are the Ising spins $\sigma_i$, which by definition can take only two values: +1 or −1. The traditional form for the microscopic energy (which from now on will be called the Hamiltonian) as the function of all the Ising spins is in the following:

$$ H = - \sum_{<i,j>} J_{ij} \sigma_i \sigma_j - h \sum_i \sigma_i $$

Here the notation $<i,j>$ indicates the summation over all the lattice sites of the nearest neighbors, $J_{ij}$ are the values of the spin–spin interactions, and $h$ is the external magnetic field. If all the $J_{ij}$s are equal to a positive constant, then one gets the ferromagnetic Ising model, and if all the $J_{ij}$s are equal to a negative constant, then one gets the antiferromagnetic Ising model.

In spite of the apparent simplicity of the Ising model, an exact solution (which means the calculation of the partition function and the correlation
1.2 The mean-field approximation

functions) has been found only for the one- and the two-dimensional systems in the zero external magnetic field. In all other cases one needs to use approximate methods. One of the simplest methods is called the mean-field approximation. In many cases this method gives results that are not too far from the correct ones, and very often it is possible to get some qualitative understanding of what is going on in the system under consideration.

The starting point of the mean-field approximation is the assumption about the structure of the probability distribution function. It is assumed that the distribution function in the equilibrium state can be factorized as the product of the independent distribution functions in the lattice sites:

\[ P[\sigma] = \frac{1}{Z} \exp(-\beta H[\sigma]) \approx \prod_i P_i(\sigma_i) \]  
(1.15)

The normalized site distribution functions take the form:

\[ P_i(\sigma_i) = \frac{1 + \phi_i}{2} \delta(\sigma_i - 1) + \frac{1 - \phi_i}{2} \delta(\sigma_i + 1) \]  
(1.16)

where \( \phi_i \) are the parameters that have to be specified.

The factorization of the distribution function, Eq. (1.15), means that the average of any product of any functions at different sites is also factorizing on the product of the independent averages:

\[ \langle f(\sigma_i)g(\sigma_j) \rangle = \langle f(\sigma_i) \rangle \langle g(\sigma_j) \rangle \]  
(1.17)

where, according to the ansatz (1.15):

\[ \langle f(\sigma_i) \rangle = \frac{1 + \phi_i}{2} f(1) + \frac{1 - \phi_i}{2} f(-1) \]  
(1.18)

In particular, for the average site magnetizations, one easily gets:

\[ \langle \sigma_i \rangle = \phi_i \]  
(1.19)

Therefore, the physical meaning of the parameters \( \{ \phi_i \} \) in the trial distribution function is that they describe the average site spin magnetizations. According to the general principles of the statistical mechanics, these parameters must be such that they would minimize the free energy of the system.

Using Eqs. (1.15) and (1.16) for the entropy and for the average energy, one gets:

\[ S = -\langle \log(P[\sigma]) \rangle \approx - \sum_i \langle \log(P_i(\sigma_i)) \rangle \]
\[ = - \sum_i \left[ \frac{1 + \phi_i}{2} \log \left( \frac{1 + \phi_i}{2} \right) + \frac{1 - \phi_i}{2} \log \left( \frac{1 - \phi_i}{2} \right) \right] \]  
(1.20)
Introduction

\[ E = -\frac{1}{2} \sum_{<i,j>} J_{ij} \phi_i \phi_j - h \sum_i \phi_i \]  \hspace{1cm} (1.21)

For the free energy, Eq. (1.10), one obtains:

\[ F = -\frac{1}{2} \sum_{<i,j>} J_{ij} \phi_i \phi_j - h \sum_i \phi_i + T \sum_i \left[ \frac{1 + \phi_i}{2} \log \left( \frac{1 + \phi_i}{2} \right) + \frac{1 - \phi_i}{2} \log \left( \frac{1 - \phi_i}{2} \right) \right] \]  \hspace{1cm} (1.22)

To be more specific, consider the ferromagnetic system on the \( D \)-dimensional cubic lattice. In this case all the spin–spin couplings are equal to some positive constant: \( J_{ij} = (1/2D)J > 0 \), (the factor \( 1/2D \) is inserted just for convenience) and each site has \( 2D \) nearest neighbors. Since the system is homogeneous, it is natural to expect that all the \( \phi \)'s must be equal to some constant \( \phi \). Then, for the free energy (1.22) one gets:

\[ \frac{F}{V} \equiv f(\phi) = \frac{1}{2} J \phi^2 - h \phi + T \left[ \frac{1 + \phi}{2} \log \left( \frac{1 + \phi}{2} \right) + \frac{1 - \phi}{2} \log \left( \frac{1 - \phi}{2} \right) \right] \]  \hspace{1cm} (1.23)

where \( V \) is the total number of sites (which is proportional to the volume of the system) and \( f \) is the density of the free energy. The necessary condition for the minimum of \( f \) is

\[ \frac{df(\phi)}{d\phi} = 0 \]

or:

\[ -J \phi - h + T \tanh(\phi) = 0 \]  \hspace{1cm} (1.24)

The resulting equation, which defines the order parameter \( \phi \) is:

\[ \phi = \tanh[\beta(J\phi + h)] \]  \hspace{1cm} (1.25)

Note, that the minimum of the free energy is conditioned by \( d^2 f / d\phi^2 > 0 \). Using Eq. (1.24), this condition can be reduced to

\[ \frac{1}{1 - \phi^2} > \beta J \]  \hspace{1cm} (1.26)

Consider first the case of a zero external magnetic field \( (h = 0) \). One can easily see that if \( T > T_c = J \), the only solution of the Eq. (1.25) \( \phi = \tanh(\beta J\phi) \) is \( \phi = 0 \), and this solution satisfies the condition (1.26). Therefore, at all temperatures higher than \( T_c \) the minimum of the free energy is achieved in the state in which all the site spin magnetizations are zero.
1.2 The mean-field approximation

Fig. 1.1. Free energy of the ferromagnetic Ising magnet: (a) in the zero external magnetic field; (b) in non-zero magnetic field.

However, if $T < T_c$, then in addition to the solution $\phi = 0$, Eq. (1.25) (with $h = 0$) has two non-trivial solutions $\phi = \pm \phi(T) \neq 0$. One can easily check that in this temperature region the solution $\phi = 0$ becomes the maximum and not the minimum of the free energy, while the true minima are achieved at $\phi = \pm \phi(T)$. Therefore, in the low-temperature region $T < T_c$ the free energy has two minima, which are characterized by non-zero site magnetizations with opposite signs.

Near $T_c$ the magnetization $\phi(T)$ is small. In this case the expansion in powers of $\phi$ in Eq. (1.25) can be made. In the leading order in $\tau \equiv (T/T_c - 1)$, $|\tau| \ll 1$, one gets:

$$\phi(T) = \text{const} |\tau|^{1/2}, \quad (\tau < 0) \quad (1.27)$$

Thus, as $T \to T_c$ from below, $\phi(T) \to 0$. The expansion of the free energy Eq. (1.23) as the function of a small value of $\phi$ yields:

$$f(\phi) = \frac{1}{2} \tau \phi^2 + \frac{1}{4} g \phi^4 - h \phi \quad (1.28)$$

where $g = T/3$ and for simplicity we have taken $J = 1$. The qualitative shapes of $f(\phi)$ at $T > T_c$ ($\tau > 0$) and at $T < T_c$ ($\tau < 0$) are shown in Fig. 1.1. Note, that since the total free energy $F$ is proportional to the volume of the system, the value of the free energy barrier separating the states with $\phi = \pm \phi(T)$ at $T < T_c$ is also proportional to the volume of the system. Therefore, in the thermodynamic limit $V \to \infty$ (which corresponds to the consideration of the macroscopic systems) the barrier separating the two states is becoming infinite.

The simple considerations described above demonstrate on a qualitative level the fundamental phenomenon called spontaneous symmetry breaking. At the temperature $T = T_c$, a phase transition of the second order occurs,
such that in the low-temperature region $T < T_c$ the symmetry with respect to the global change of the signs of the spins is broken, and two (instead of one) ground states appear. These two states differ by the sign of the average spin magnetization, and they are separated by the macroscopic barrier of the free energy.

1.3 Quenched disorder, selfaveraging and the replica method

In this book we will consider the thermodynamical properties of various spin systems that are characterized by the presence of some kind of a quenched disorder. In realistic magnetic materials such disorder can exist, for example, owing to the oscillating nature of the exchange spin–spin interactions combined with the randomness in the positions of the interacting spins (such as in metallic spin-glass alloys AgMn), or owing to defects in the lattice structure, or because of the presence of impurities, etc.

Because we will mostly be interested in the qualitative effects produced by the quenched disorder, the details of the realistic structure of such magnetic systems will be left aside. Here we will concentrate on extremely simplified model description of the disordered spin systems.

In what follows we will consider two essentially different types of disordered magnet. First, we will study the thermodynamic properties of spin systems in which the disorder is strong. The term ‘strong disorder’ refers to the situation when the disorder appears to be the dominant factor for the ground-state properties of the system, so that it dramatically changes the low-temperature properties of the magnetic system as compared with the usual ferromagnetic phase. These types of system, usually called the spin glasses, will be considered in the first part of the course.

In the second part of the course we will consider the properties of weakly disordered magnets. This is the case when the disorder does not produce notable effects for the ground-state properties. It will be shown, however, that in certain cases even a small amount of disorder can produce dramatic effects for the critical properties of the system in close vicinity of the phase transition point.

The main problem in dealing with disordered systems is that the disorder here is quenched. Formally, all the results for the observable quantities for a given concrete system must depend on the concrete interaction matrix $J_{ij}$, i.e. these results are defined by a macroscopic number of random parameters. Apparently, results of this type are impossible to calculate and, moreover, they are useless. Intuitively it is clear, however, that the quantities
1.3 Quenched disorder, selfaveraging and the replica method

which are called the observables should depend on some general averaged characteristics of the random interactions. This brings us to the concept of selfaveraging.

The traditional speculation about why the selfaveraging phenomenon should be expected to take place, is as follows. The free energy of the system is known to be proportional to the volume \( V \) of the system. Therefore, in the thermodynamic limit \( V \to \infty \) the main contribution to the free energy comes from the volume, and not from the boundary, which usually produces effects of the next orders in the small parameter \( 1/V \). Any macroscopic system could be divided into a macroscopic number of macroscopic subsystems. Then the total free energy of the system would consist of the sum of the free energies of the subsystems, plus the contribution that comes from the interactions of the subsystems, at their boundaries. If all the interactions in the system are short range (which takes place in any realistic system), then the contributions from the mutual interactions of the subsystems are just the boundary effects, which vanish in the thermodynamic limit. Therefore, the total free energy could be represented as a sum of the macroscopic number of terms. Each of these terms is a random quenched quantity because it contains, as parameters, the elements of the random spin–spin interaction matrix. In accordance with the law of large numbers, the sum of many random quantities can be represented as their average value, obtained from their statistical distribution, multiplied by their number (all this is true, of course, only under certain requirements on the characteristics of the statistical distribution). Therefore, the total free energy of a macroscopic system must be selfaveraging over the realizations of the random interactions in accordance with their statistical distribution.

The free energy is known to be given by the logarithm of the partition function. Thus, in order to calculate the observable thermodynamics, one has to average the logarithm of the partition function over the given distribution of random \( J_{ij} \)s after the calculation of the partition function itself. To perform such a program the following technical trick, which is called the replica method, is used.

Formally, the replicas are introduced as follows. In order to obtain the physical (selfaveraging) free energy of the quenched random system we have to average the logarithm of the partition function:

\[
F = \overline{\ln Z} = -\frac{1}{\beta} \overline{\ln(Z_{\beta})} \tag{1.29}
\]

where \(\overline{\ldots} \) denotes the averaging over random interactions \( \{J_{ij}\} \) with a given
distribution function $P[J]$:

$$\langle \ldots \rangle = \left( \prod_{<i,j>} \int dJ_{ij} \right) P[J] \langle \ldots \rangle$$

(1.30)

and the partition function is

$$Z_J = \sum_{\sigma} \exp(-\beta H[J, \sigma])$$

(1.31)

To perform this averaging procedure, the following trick is invented. Let us consider the integer power $n$ of the partition function (1.31). This quantity is the partition function of the $n$ non-interacting identical replicas of the original system (i.e. having identical fixed spin–spin couplings $J_{ij}$):

$$Z^n_J = \left( \prod_{a=1}^{n} \sum_{\sigma^a} \right) \exp \left( -\beta \sum_{a=1}^{n} H[J, \sigma_a] \right)$$

(1.32)

Here the subscript $a$ labels the replicas. Let us introduce the quantity:

$$F_n = -\frac{1}{\beta n} \ln(Z_n)$$

(1.33)

where

$$Z_n = Z^n_J$$

(1.34)

Now, if a formal limit $n \to 0$ is taken in the expression (1.33), then the original expression for the physical free energy (1.29) will be recovered:

$$\lim_{n \to 0} F_n = -\lim_{n \to 0} \frac{1}{\beta n} \ln(Z_n) = -\lim_{n \to 0} \frac{1}{\beta n} \ln \left[ \exp(n \ln Z_J) \right] = -\frac{1}{\beta} \ln Z_J = F$$

(1.35)

Thus, the scheme of the replica method can be described in the following steps. First, the quantity $F_n$ for the integer $n$ must be calculated. Second, the analytic continuation of the obtained function of the parameter $n$ should be made for an arbitrary non-integer $n$. Finally, the limit $n \to 0$ has to be taken. Although this procedure may look rather doubtful at first, actually it is not so crazy! First, if the free energy appears to be an analytic function of the temperature and the other parameters (so that it can be represented as the series in powers of $\beta$), then the replica method can be easily proved to be correct in a strict sense. Second, in all cases, when the calculations can be performed by some other method, the results of the replica method are confirmed.

One could also introduce replicas another way [2, 29, 30]. Let us consider a general spin system described by a Hamiltonian $H[J; \sigma]$, which depends