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Classical scattering

Almost all we see and perceive comes to us indirectly by the scattering of light from various objects; that is, by the scattering of electromagnetic radiation over a very restricted interval of the frequency spectrum. Much of this merely illuminates our world and helps us move about, while some exceptional natural scattering phenomena such as rainbows, glories, and halos touch our aesthetic sense. On a technical level, a very large portion of what we have learned about the physical world over the past four millennia has come to us via scattering experiments with both particles and waves, so that a study of scattering theory is an integral part of physics itself.

Classically the most familiar type of scattering is that among particles, such as balls on a pool table – or, more deeply, among gas molecules in the room where we work. Equally evident, however, are the results of scattering of electromagnetic and sound waves, and at first these appear to be entirely different phenomena. Just as modern quantum theory has compelled us to view all matter in terms of a particle–wave dichotomy, however, so have we also learned to view scattering processes as both particle-like and wave-like. That is, at high frequencies and short wavelengths even intrinsically wave-like classical phenomena exhibit particle-like scattering behavior, whereas on the quantum level particle scattering usually must be viewed in terms of waves.

A common experience is to find oneself in a crowded auditorium listening to a speaker who is shielded from view by a pillar, say. While you can hear the speaker perfectly well, you are unable to see him. Although the signals are both transmitted via waves, the wavelength of sound ($\simeq 1\,\mathrm{m}$) is very much greater than that of the scattered light ($\simeq 10^{-7}\,\mathrm{m}$), so that the latter scatters more like a particle, whereas the former wave is able to 'bend' around the pillar while maintaining the correlation of density fluctuations. Almost all physical phenomena exhibit this form of 'complementarity' on one scale or another, and which particular view provides the most useful description in any situation is governed by some combination of the wavelength and the scattering geometry. It will be found useful in what follows to classify scattering behavior in terms of three general domains.

The classical domain: particle and particle-like trajectories; geometric optics.

The wave domain: pure quantum mechanics; pure acoustic and electromagnetic waves; physical optics.

The semiclassical domain: the vast intermediate region between the above two, containing many interesting physical phenomena.

There is a strong analogy here to our understanding of the principal phases of matter, wherein gases are dominated by kinetic energy and the notion of free particles at high temperatures and low densities. The opposite domain of low temperatures and high densities is characterized by the crystal lattice and potential-energy dominance. In the large region between these two extremes lies the liquid state, whose description is much more difficult owing to the equal importance of kinetic energy and potential energy. Similarly, the semiclassical domain is that of intermediate wavelengths in which construction of suitable approximations presents greater difficulties. Nevertheless, it is also a region containing a great deal of interesting physics, and will receive a large measure of attention in that which follows.

Our primary interest in the following chapters will focus on the scattering of classical waves, both scalar and vector, from spherical targets. Specifically, quantum-mechanical scattering will not be discussed at any length, though frequent reference to that theory will be made where it is useful to elucidate various points in the classical theory. Spherical symmetry provides a natural basic model for which the mathematics is entirely tractable, yet corresponds closely to reality over a large range of phenomena.† It introduces considerable simplicity without sacrificing the physics, for only three fundamental parameters are required to characterize the scattering process: the target radius a, the incident wavelength λ, and the index of refraction n of the sphere – geometry, degree of wave-like behavior, and material. Let us begin with the familiar classical domain.

1.1 Particle scattering

Although we shall be interested primarily in the scattering of waves in that which follows, it will be useful to begin by reviewing some elementary aspects of classical particle scattering. This serves not only to affirm common notation, but also to re-introduce geometric arrangements and some fundamental physical ideas that will remain valid throughout the subsequent

† Nonspherical targets will be discussed to some extent in Chapter 8.

discussion. In practice the experimenter usually directs a beam of particles at a distribution of targets and arranges an array of detectors at asymptotic distances to record the distribution of scattered particles. Almost always it is safe to assume that neither the particles in the beam nor the scattered particles interact significantly with one another. In addition, we ignore multiple scattering so that we can focus on the truly essential aspects of the process by studying only a single particle scattering from a single target.

When two particles scatter from one another in the rest frame of the observer the process is said to take place in the *laboratory frame*. If both linear momentum and translational kinetic energy are conserved the collision is called *elastic* and we can write

$$\boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{p}_1' + \boldsymbol{p}_2', \tag{1.1a}$$

$$T_1 + T_2 = T_1' + T_2', \tag{1.1b}$$

where $T \equiv p^2/2m$, and primed variables here denote scattered quantities. Occasionally it is necessary to conserve explicitly angular momenta as well. These conservation laws provide four equations in 13 unknowns: 12 components of momentum and the ratio of the two masses. In one way or another, then, one must determine or specify nine variables to solve the scattering problem completely.

Consider now the example of particle m_1 incident with momentum $\boldsymbol{p}_1$ upon particle m_2 initially at rest, as in Fig. 1.1. Momentum conservation demands that $\boldsymbol{p}_2'$ be in the plane defined by $\boldsymbol{p}_1$ and $\boldsymbol{p}_1'$, so that the problem is effectively two-dimensional. If m_1, m_2, and $\boldsymbol{p}_1$ are known, and θ_1 is measured, then Eqs. (1.1) yield

$$p_1 = p_1' \cos\theta_1 + p_2' \cos\theta_2, \tag{1.2a}$$

$$p_2'^{\,2} = p_1^2 + p_1'^{\,2} - 2p_1 p_1' \cos\theta_1, \tag{1.2b}$$

$$\frac{p_1'}{p_1} = m_1 \frac{\cos\theta_1}{m_1 + m_2} \pm \left(\frac{m_1^2 \cos^2\theta_1}{(m_1+m_2)^2} + \frac{m_2 - m_1}{m_1 + m_2} \right)^{1/2}. \tag{1.2c}$$

These three equations are then solved for p_1', p_2', and θ_2, so that if the masses and initial momenta are known the problem is solved completely by measuring the *scattering angle* θ_1. One could also focus on the recoil angle θ_2, but it is usually θ_1 that is measured.

Note that, for elastic scattering and $m_1 > m_2$, the radical in Eq. (1.2c) implies the existence of a maximum scattering angle

$$\theta_m \equiv |\theta_1|_{\max} = \sin^{-1}(m_2/m_1), \tag{1.3}$$

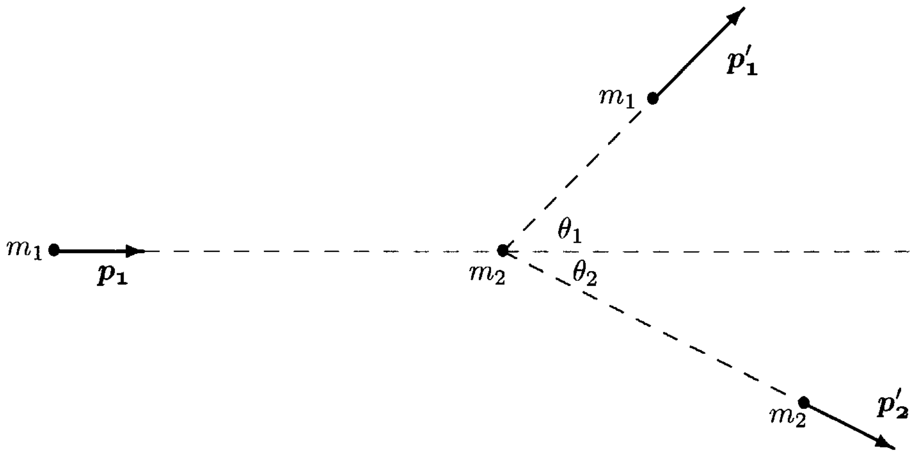

Fig. 1.1. Two-body scattering in the laboratory.

such that $0 \leq \theta_m \leq \pi/2$. However, if $m_1 \leq m_2$, then only the plus sign in Eq. (1.2c) is physical and $0 \leq \theta_1 \leq \pi$ always.

To extend the discussion to relativistic scattering one need only replace $\boldsymbol{p}$ and T by their relativistic counterparts, but this scenario will not be treated further here. If the collision is *inelastic*, then some kinetic energy is either absorbed (an endoergic process) or released (an exoergic process) in the process. Because of their great diversity such processes are difficult to treat in complete generality, but conservation of energy still leads to a definite set of equations to be solved, given sufficient initial data. In later chapters we shall find it relatively simple to account for energy loss in the scattering of waves by absorptive targets.

When Newton's third law is valid, as is usually the case, the two-body system can be reduced to an equivalent one-body problem. In the laboratory frame the equations of motion are

$$m_1\ddot{\boldsymbol{r}}_1 = \boldsymbol{F}_1, \qquad m_2\ddot{\boldsymbol{r}}_2 = \boldsymbol{F}_2, \tag{1.4}$$

where dots denote time derivatives, $\boldsymbol{F}_1 = -\boldsymbol{F}_2$, and we shall not consider any external forces to be present. Now introduce relative and center-of-mass coordinates, respectively:

$$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2,$$
$$\boldsymbol{R} = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1 + m_2}. \tag{1.5}$$

With the definitions of total and reduced masses,

$$M \equiv m_1 + m_2, \qquad \mu \equiv m_1 m_2 / M, \tag{1.6}$$

respectively, the equations of motion (1.4) become

$$M\ddot{\boldsymbol{R}} = 0, \qquad \mu\ddot{\boldsymbol{r}} = \boldsymbol{F}_1, \tag{1.7}$$

thereby separating the relative from the center-of-mass motion.

In like manner, one can define center-of-mass and relative velocities, respectively, as

$$\begin{aligned} \boldsymbol{V} &\equiv \dot{\boldsymbol{R}} = \frac{m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2}{M}, \\ \boldsymbol{v} &\equiv \dot{\boldsymbol{r}} = \boldsymbol{v}_1 - \boldsymbol{v}_2. \end{aligned} \tag{1.8}$$

In terms of these parameters the kinetic energy, angular momentum, and linear momentum of the system can be written, respectively, as

$$T = \tfrac{1}{2}MV^2 + \tfrac{1}{2}\mu v^2, \tag{1.9a}$$

$$\boldsymbol{L} = M(\boldsymbol{R} \times \boldsymbol{V}) + \mu(\boldsymbol{r} \times \boldsymbol{v}), \tag{1.9b}$$

$$\boldsymbol{P} = m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2 = M\boldsymbol{V}. \tag{1.9c}$$

The absence of any contribution of the form $\mu\boldsymbol{v}$ to the total linear momentum suggests a very useful coordinate transformation obtained by inverting Eqs. (1.5):

$$\begin{aligned} \boldsymbol{r}_1 &= \boldsymbol{R} + \frac{m_2}{M}\boldsymbol{r}, \\ \boldsymbol{r}_2 &= \boldsymbol{R} - \frac{m_1}{M}\boldsymbol{r}. \end{aligned} \tag{1.10}$$

In the *center-of-mass* (CM) coordinate system $(\boldsymbol{r}_1 - \boldsymbol{R}, \boldsymbol{r}_2 - \boldsymbol{R})$ the motion is thus described entirely in terms of the relative coordinate $\boldsymbol{r}$ of a fictitious particle with mass μ – effectively a one-body problem. Furthermore, in this system the total momentum is zero. Alternatively, from Eqs. (1.5) we see that this is simply a transformation to that system in which $\boldsymbol{V}$ is zero.

Figure 1.2 illustrates an actual one-body problem in which a particle of mass m is scattered from a repulsive fixed target or scattering center. The quantity b is called the *impact parameter* and measures the distance of the asymptotic path of the incoming particle from that of a head-on collision, while θ is again the scattering angle. Although we think of this process as taking place in the laboratory system, it is clearly equivalent to 2-body scattering with mass μ in the CM system. This equivalence becomes especially useful when we move from the Newtonian formulation to one in terms of the Lagrangian or Hamiltonian, whereby the particle interaction is described

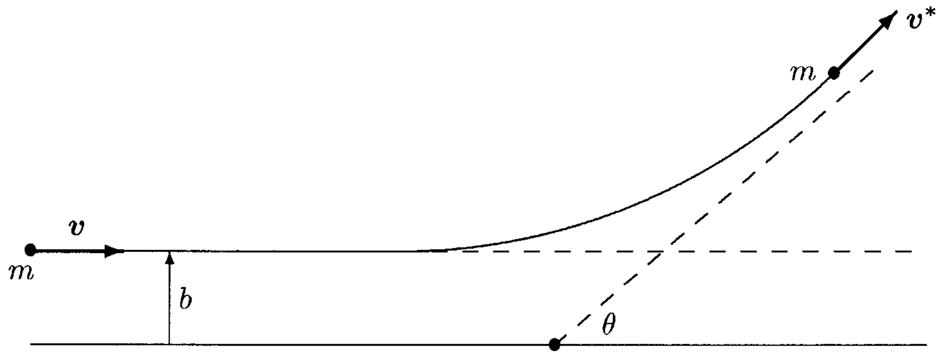

Fig. 1.2. A schematic illustration of either an effective or an actual one-body scattering process, in which a particle is incident with impact parameter b on a fixed scattering center.

by a central potential $V(r) \equiv V(|\boldsymbol{r}_2 - \boldsymbol{r}_1|)$. The potential energy is the same in either scenario, for it depends only on the relative coordinate $|\boldsymbol{r}|$. In this one-body problem the scattering is described completely by conservation of angular momentum $\boldsymbol{L}$ and energy E,

$$L = |\boldsymbol{L}| = mr^2\dot{\varphi}, \qquad E = \tfrac{1}{2}m\dot{r}^2 + U_L(r), \tag{1.11}$$

where φ is the polar angle in the scattering plane, and U_L is the *effective potential*

$$U_L(r) = V(r) + \frac{L^2}{2mr^2}, \tag{1.12}$$

a sum of the interaction and centrifugal potentials. In our subsequent investigations it will suffice to focus only on this one-body scenario.

Figure 1.2 indicates that the radial variable ranges from $r = \infty$ to the largest root r_0 of $\dot{r} = 0$, called the *classical distance of closest approach.* Similarly, φ appears to range from π at $r = \infty$ to the scattering angle θ as r again recedes to infinity. However, the latter conclusion is not true in general, as is readily seen by integrating $\dot{\varphi}/\dot{r} = d\varphi/dr$ in Eq. (1.11). The result is called the *deflection angle* Θ,

$$\Theta(L) = \pi - 2L \int_{r_0}^{\infty} \frac{dr}{r^2\sqrt{2m[E - U_L(r)]}}, \tag{1.13}$$

and ranges over $(0, \pi)$ for a repulsive (positive) interaction. But for an attractive (negative) interaction the particle can go around the scattering center any number of times before emerging in the direction of the scattering angle, so that the two are related by

$$\Theta + 2n\pi = \pm\theta, \tag{1.14}$$

where n is a nonnegative integer such that θ remains within its physical range $(0, \pi)$. Since $|\boldsymbol{L}| = b|\boldsymbol{p}|$, in terms of the asymptotic linear momentum $\boldsymbol{p}$, we see that Θ is equivalently a function of the impact parameter.

As mentioned earlier, scattering experiments generally involve a beam of particles incident upon a target, so that the particles are distributed more or less uniformly over the azimuthal angle ϕ around the incident z-axis. We define the *effective scattering cross section* $d\sigma$ as the ratio of the number of particles scattered per unit time between angles θ and $\theta + d\theta$ and the number of particles incident per unit time through unit beam cross section. Hence,

$$d\sigma = 2\pi b|db|. \tag{1.15}$$

We are actually interested in the number of outgoing particles in a solid angle $d\Omega = \sin\theta\, d\theta d\phi$, so that the quantity of experimental importance is the *differential scattering cross section*:

$$\frac{d\sigma}{d\Omega} = \frac{b(\theta)}{\sin\theta}\left|\frac{db}{d\Theta}\right|, \tag{1.16}$$

which must necessarily be positive. Note that $d\Theta$ can be replaced by $d\theta$ in this equation, so that the *total scattering cross section* is obtained by integrating over all solid angles:

$$\sigma = \int \left(\frac{d\sigma}{d\Omega}\right) d\Omega. \tag{1.17}$$

If the scattering potential is attractive, or has an attractive portion, then the particle can go around the center many times and $b(\theta)$ is a many-valued function of θ. That is, many different impact parameters can lead to the same scattering angle. Consequently, Eq. (1.16) should be generalized to a sum over all impact parameters leading to a given θ:

$$\frac{d\sigma}{d\Omega} = \sum_j \frac{b_j(\theta)}{\sin\theta}\left|\frac{d\theta}{db_j}\right|^{-1}. \tag{1.18}$$

This expression suggests a number of very interesting phenomena that can arise in processes of this type, to which we shall return presently.

Some simple examples

We shall present four very simple examples of pure classical particle scattering that contain some useful ideas as well as providing fundamental models.† Consider first a particle of mass m with velocity $\boldsymbol{v}_1$ in a half-space in which its constant potential energy is V_1. It leaves this half-space and enters another in which the potential energy is some other constant V_2, and we wish to determine the final direction of the particle. Let θ_1 and θ_2 be the respective angles $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ make with the normal to the plane separating the two half-spaces before and after penetrating the plane. Because the potential energy is independent of coordinates along this plane, linear momentum must be conserved in those directions. In particular, $v_1 \sin\theta_1 = v_2 \sin\theta_2$, and the relation between v_1 and v_2 is determined by energy conservation. Hence,

$$\frac{\sin\theta_1}{\sin\theta_2} = \sqrt{1 + \frac{2}{mv_1^2}(V_1 - V_2)}. \tag{1.19}$$

We shall see that this is a result reminiscent of, but not equivalent to, Snell's law in geometric optics.

A second example involves the spherical target of Fig. 1.3, an impenetrable sphere of radius a defined such that the interaction with the incident particle is

$$V(r) = \begin{cases} \infty, & r < a \\ 0, & r > a. \end{cases} \tag{1.20}$$

Owing to the impenetrability of the particle, its path is a pair of straight lines symmetric about a radius to the point of impact. Clearly, $b = a\sin\phi$ and the scattering angle is $\theta = \pi - 2\phi$. The differential scattering cross section is then

$$\frac{d\sigma}{d\Omega} = \tfrac{1}{4}a^2, \tag{1.21}$$

and an integration over all solid angles gives the total cross section as $\sigma = \pi a^2$, the geometric cross section of the sphere. This quantity is just the cross section removed from the incident beam and implies the existence of a circular-cylindrical 'shadow' of radius a extending to infinity to the right of the sphere.

Our third example extends the above model to the transparent sphere of Fig. 1.4, which is basically a potential well of radius a and depth V_0:

$$V(r) = \begin{cases} V = -V_0, & r < a \\ 0, & r > a. \end{cases} \tag{1.22}$$

† The first three are taken from Landau and Lifshitz (1960).

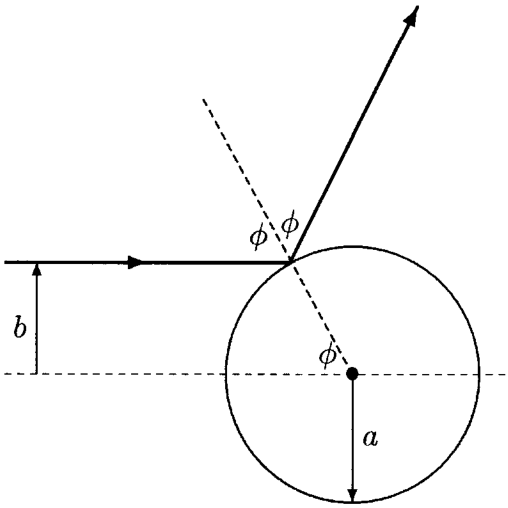

Fig. 1.3. Particle scattering from an impenetrable, or hard, sphere.

From Eq. (1.19) we see that

$$\frac{\sin\alpha}{\sin\beta} = \sqrt{1 + 2V_0/(2mv^2)} \equiv n, \tag{1.23}$$

where $\boldsymbol{v}$ is the incident velocity. From Fig. 1.4, $\theta = 2(\alpha - \beta)$, and $b = a\sin\alpha$. Elimination of α from these two equations provides the relation between b and θ,

$$b^2 = \frac{a^2n^2\sin^2(\theta/2)}{n^2 + 1 - 2n\cos(\theta/2)}, \tag{1.24}$$

and substitution into Eq. (1.16) yields

$$\frac{d\sigma}{d\Omega} = \frac{a^2n^2}{4\cos(\theta/2)}\frac{[n\cos(\theta/2) - 1]\,[n - \cos(\theta/2)]}{[n^2 + 1 - 2n\cos(\theta/2)]^2}. \tag{1.25}$$

The scattering angle varies from 0 at $b = 0$ to a maximum value θ_{m} at $b = a$ determined by $\cos(\theta_{\mathrm{m}}/2) = 1/n$. The total cross section is again πa^2.

As a final example, consider the quintessentially long-range potential having the Coulomb or gravitational form: $V(r) = \alpha r^{-1}$. The integral in Eq. (1.13) is evaluated exactly in this case, yielding $\Theta = 2\tan^{-1}[\alpha/(2Eb)]$. Then, solving this for b, we find for the differential cross section the well-known Rutherford expression

$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha}{4E\sin^2(\theta/2)}\right)^2. \tag{1.26}$$

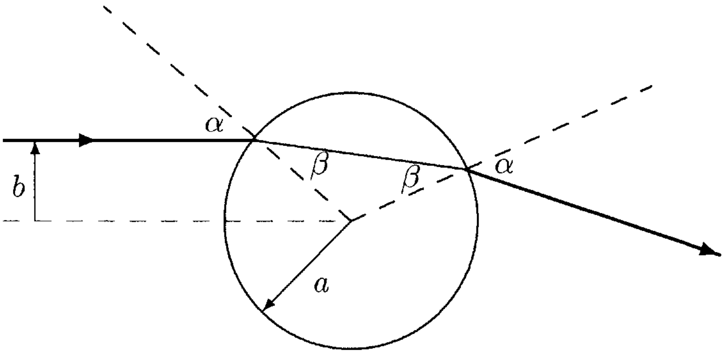

Fig. 1.4. Particle scattering from a transparent, or soft, sphere.

The characteristic divergence in the forward direction is quite evident here.

Singular effects in classical scattering

We now return to Eq. (1.18) and consider the family of effective potentials illustrated in Fig. 1.5, parametrized by the incident angular momentum (or impact parameter). Suppose that $U_L(r)$ achieves a local maximum for $L = L_0$ at $r = r_0$ and energy E. The particle enters an unstable orbit at this point and the integral in Eq. (1.13) diverges logarithmically at the lower limit. The particle will then spiral indefinitely around the scattering center, so that $b(\theta)$ becomes *infinitely* many-valued. This phenomenon of *orbiting*, or *spiral scattering*, appears to have first been discussed extensively by Hirschfelder *et al.* (1954). For $L < L_0$ a potential well develops for $r < r_0$, so that a particle with energy below the top of the barrier finds itself in an oscillating orbit within the well.

The differential cross section of Eq. (1.16) possesses an evident singularity whenever the deflection function passes through a maximum or a minimum:

$$\left(\frac{d\Theta}{db}\right)_{\theta=\theta_R} = 0, \tag{1.27}$$

where θ_R is called the *rainbow angle*. The analogy with the optical rainbow is suggested by the obvious clustering of a large number of particles at a stationary point of Θ, an analogy that will become much clearer in Chapter 4. If the potential is everywhere attractive, meaning that it is negative and monotonically increasing, then Θ vanishes as $b \to 0$. This is also true if the potential is positive, but E is greater than the local maximum. Since Θ must