CHAPTER ONE

Introduction

The propagation of seismic body waves in complex, laterally varying 3-D layered structures is a complicated process. Analytical solutions of the elastodynamic equations for such types of media are not known. The most common approaches to the investigation of seismic wavefields in such complex structures are (a) methods based on direct numerical solutions of the elastodynamic equation, such as the finite-difference and finite-element methods, and (b) approximate high-frequency asymptotic methods. Both methods are very useful for solving certain types of seismic problems, have their own advantages and disadvantages, and supplement each other suitably.

We will concentrate here mainly on high-frequency asymptotic methods, such as the ray method. The high-frequency asymptotic methods are based on an asymptotic solution of the elastodynamic equation. They can be applied to compute not only rays and travel times but also the ray-theory amplitudes, synthetic seismograms, and particle ground motions. These methods are well suited to the study of seismic wavefields in smoothly inhomogeneous 3-D media composed of thick layers separated by smoothly curved interfaces. The high-frequency asymptotic methods are very general; they are applicable both to isotropic and anisotropic structures, to arbitrary 3-D variations of elastic parameters and density, to curved interfaces arbitrarily situated in space, to an arbitrary source-receiver configuration, and to very general types of waves. High-frequency asymptotic methods are also appropriate to explain typical "wave" phenomena of seismic waves propagating in complex 3-D isotropic and anisotropic structures. The amplitudes of seismic waves calculated by asymptotic methods are only approximate, but their accuracy is sufficient to solve many 3-D problems of practical interest.

Asymptotic high-frequency solutions of the elastodynamic equation can be sought in several alternative forms. In the ray method, they are usually sought in the form of the so-called *ray series* (see Babich 1956; Karal and Keller 1959). For this reason, the ray method is also often called the *ray-series method*, or the *asymptotic ray theory* (ART).

The seismic ray method can be divided into two parts: *kinematic* and *dynamic*. The kinematic part consists of the computation of seismic rays, wavefronts, and travel times. The dynamic part consists of the evaluation of the vectorial complex-valued amplitudes of the displacement vector and the computation of synthetic seismograms and particle ground motion diagrams.

The most strict approach to the investigation of both kinematic and dynamic parts of the ray method consists of applying asymptotic high-frequency methods to the elastodynamic equations. The kinematic part of the ray method, however, may also be attacked by some simpler approaches, for example, by variational principles (Fermat principle). It is even possible to develop the whole kinematic part of the seismic ray method using the well-known Snell's law. Such approaches have been used for a long time in seismology and have given a number of valuable results. There may be, however, certain methodological objections to their application. In the application of *Snell's law*, we must start from a model consisting of homogeneous layers with curved interfaces and pass from this model to a smoothly varying model by increasing the number of interfaces. Such a limiting process offers very useful seismological insights into the ray tracing equations and travel-time computations in inhomogeneous media, but it is more or less intuitive. The *Fermat principle* has been used in seismology as a rule independently for P and S waves propagating in inhomogeneous media. The elastic wavefield, however, can be separated into P and S waves only in homogeneous media (and perhaps in some other simple structures). In laterally varying media with curved interfaces, the wavefield is not generally separable into P and S waves; the seismic wave process is more complicated. Thus, we do not have any exact justification for applying the principle independently to P and S waves. In media with larger velocity gradients, the ray method fails due to the strong coupling of P and S waves. Only the approach based on the asymptotic solution of the elastodynamic equation gives the correct answer: the separation of the seismic wavefield in inhomogeneous media into two independent wave processes (P and S) is indeed possible, but it is only *approximate, in that it is valid for high frequencies and sufficiently smooth media only*.

Similarly, certain properties of vectorial complex-valued amplitudes of seismic body waves can be derived using energy concepts, particularly using the expressions for the energy flux. Such an approach is again very useful for intuitive physical understanding of the amplitude behavior, but it does not give the complete answer. The amplitudes of seismic body waves have a vectorial complex-valued character. The waves may be elliptically polarized (S waves) and may include phase shifts. These phase shifts influence the waveforms. The energy principles do not yield a complete answer in such situations. Consequently, they cannot be applied to the computation of synthetic seismograms and particle ground motion diagrams.

Recently, several new concepts and methods have been proposed to increase the possibilities and efficiency of the standard ray method; they include *dynamic ray tracing, the ray propagator matrix*, and *paraxial ray approximations*. In the standard ray method, the travel time and the displacement vector of seismic body waves are usually evaluated along rays. Thus, if we wish to evaluate the seismic wavefield at any point, we must find the ray that passes through this point (boundary value ray tracing). The search for such rays sometimes makes the application of the standard ray method algorithmically very involved, particularly in 3-D layered structures. The paraxial ray methods, however, allow one to compute the travel time and displacement vector not only along the ray but also in its paraxial vicinity. It is not necessary to evaluate the ray that passes exactly through the point. The knowledge of the ray propagator matrix makes it possible to solve analytically many complex wave propagation problems that must be solved numerically by iterations in the standard ray method. This capability greatly increases the efficiency of the ray method, particularly in 3-D complex structures.

The final ray solution of the elastodynamic equation is composed of elementary waves corresponding to various rays connecting the source and receiver. Each of these elementary waves (reflected, refracted, multiply reflected, converted, and the like) is described by its own ray series. In practical seismological applications, the higher terms of the ray series have not yet been broadly used. In most cases, the numerical modeling of seismic wavefields and the interpretation of seismic data by the ray method have been based on the zeroth-order leading term of the ray series. In this book, mainly the zeroth-order terms of the ray series are considered. These zeroth-order terms, however, are treated here in a great detail. Concise expressions for the zeroth-order ray-theory Green function for a point source and receiver situated at any place in a general 3-D, layered and blocked, structure are derived. For a brief treatment of the higher-order terms of the ray series for the scalar (acoustic) and vectorial (elastic) waves see Sections 5.6 and 5.7.

As is well known, the ray method is only approximate, and its applications to certain seismological problems have some restrictions. Recently, several new extensions of the ray method have been proposed; these extensions overcome, partially or fully, certain of these restrictions. They include the method of summation of Gaussian beams, the method of summation of Gaussian wave packets, and the Maslov-Chapman method. These methods have been found very useful in solving various seismological problems, even though certain aspects of these methods are still open for future research.

The whole book may be roughly divided into five parts.

In the first part, the main principles of the asymptotic high-frequency method as it is used to solve the elastodynamic equation in a 3-D laterally varying medium are briefly explained and discussed. A particularly simple approach is used to derive and discuss the most important equations and related wave phenomena from the seismological point of view. It is shown how the elastic wavefield is approximately separated into individual elementary waves. These individual waves propagate independently in a smoothly varying structure, their travel times are controlled by the eikonal equation, and their amplitudes are controlled by the transport equation. Various important phenomena of seismic wavefields connected with 3-D lateral variations and with curved interfaces are derived and explained, both for isotropic and anisotropic media. Great attention is devoted to the differences between elastic waves propagating in isotropic and anisotropic structures. Exact and approximate expressions for acoustic and elastodynamic Green functions in homogeneous media are also derived. See Chapter 2.

The second part is devoted to ray tracing and travel-time computation in 3-D structures. The ray tracing and travel-time computation play an important role in many seismological applications, particularly in seismic inversion algorithms, even without a study of ray amplitudes, polarization, and wavelet shape. In addition to individual rays, the ray fields are also introduced in this part. The singular regions of the ray fields and related wave phenomena are explained. Special attention is devoted to the definition, computation, and physical meaning of the geometrical spreading. See Chapter 3.

The third part is devoted to dynamic ray tracing and paraxial ray methods. The paraxial ray methods can be used to compute the travel time and other important quantities not only along the ray but also in its vicinity. Concepts of dynamic ray tracing and of the ray propagator matrix are explained. The dynamic ray tracing is introduced both in ray-centered and Cartesian coordinates, for isotropic and anisotropic structures. Various important applications of the paraxial ray method are explained. See Chapter 4.

The fourth part of the book discusses the computation of ray amplitudes. Very general expressions for ray amplitudes of an arbitrary multiply reflected/transmitted (possibly converted) seismic body wave propagating in acoustic, elastic isotropic, elastic anisotropic, laterally varying, layered, and block structures are derived. The medium may also be weakly dissipative. Both the source and the receiver may be situated either in a smooth medium or at a structural interface or at the Earth's surface. Final equations for the amplitudes of the ray-theory elastodynamic Green function is laterally varying layered structures are derived. Great attention is also devoted to the ray-series solutions, both in the frequency and the time domain. The seismological applications of higher-order terms of the ray series are discussed. See Chapter 5.

The fifth part explains the computation of ray synthetic seismograms and ray synthetic particle ground motions. Several possibilities for the computation of ray synthetic seismograms are proposed: in the frequency domain, in the time domain, and by the summation of elementary seismograms. Advantages and disadvantages of individual approaches are discussed. Certain of these approaches may be used even for dissipative media. The basic properties of linear, elliptic, and quasi-elliptic polarization are described. The causes of quasi-elliptic polarization of S waves are briefly summarized. See Chapter 6.

This book, although very extensive, is still not able to cover all aspects of the seismic ray method. This would increase its length inadmissibly. To avoid this, the author has not discussed many important subjects regarding the seismic ray method (or related closely to it) or has discussed them only briefly. Nevertheless, the reader should remember that the main aim of this book is to present a detailed and complete description of the seismic ray method with a real-valued eikonal for 3-D, laterally varying, isotropic or anisotropic, layered, and block structures. The author, however, does not and had no intention of including all the extensions and applications of the seismic ray method and all the problems related closely to it. We shall now briefly summarize several important topics that are related closely to the seismic ray method but that will not be treated in this book or that will be treated more briefly than they would deserve.

1. Although the seismic ray method developed in this book plays a fundamental role in various inverse problems of seismology and of seismic exploration and in many interpretational procedures, the actual inversion and interpretational procedures are not explicitly discussed here. These procedures include seismic tomography, seismic migration, and the location of earthquake sources, among others.

2. The seismic ray method has found important applications in forward and inverse scattering problems. With the exception of a brief introduction in Section 2.6.2; the scattering problems themselves, however, are not discussed here.

3. The seismic ray method may be applied only to structural models that satisfy certain smoothness criteria. The construction of 2-D and 3-D models that would satisfy such criteria is a necessary prerequisite for the application of the seismic ray method, but is not discussed here at all. Mostly, it is assumed that the model is specified in Cartesian rectangular coordinates. Less attention is devoted to models specified in curvilinear coordinate systems (including spherical); see Section 3.5.

4. The seismic ray method developed here may be applied to high-frequency seismic body waves propagating in deterministic, perfectly elastic, isotropic or anisotropic media. Other types of waves (such as surface waves) are only briefly mentioned. Moreover, viscoelastic, poroelastic, and viscoporoelastic models are not considered. The exception is a weakly dissipative (and dispersive) model that does not require complex-valued ray tracing; see Sections 5.5 and 6.3.5. In Sections 2.6.4 and 5.6.8, the space-time ray method and the ray method with a complex eikonal are briefly discussed, even though they deserve considerably more attention. The computation of complex-valued rays in particular (for example, in dissipative media, in the caustic shadow, and in some other singular regions) may be very important in applications. Actually, the seismic ray method, without considering complex rays, is very incomplete.

5. Various extensions of the seismic ray method have been proposed in the literature. These extensions include the asymptotic diffraction theory, the method of edge waves, the method of the parabolic wave equation, the Maslov-Chapman method, and the method of summation of Gaussian beams or Gaussian wave packets, among others. Here we shall treat, in some detail, only the extensions based on the summation of paraxial ray approximations and on the summation of paraxial Gaussian beams; see Section 5.8. The method based on the summation of paraxial ray approximations yields integrals close or equal to those of the Maslov-Chapman method. The other extensions of the seismic ray method are discussed only very briefly in Section 5.9, but the most important references are given there.

6. No graphical examples of the computation of seismic rays, travel times, ray amplitudes, synthetic seismograms, and particle ground motions in 3-D complex models are presented for two reasons. First, most figures would have to be in color, as 3-D models are considered. Second, the large variety of topics discussed in this book would require a large number of demonstration figures. This would increase the length and price of the book considerably. The interested readers are referred to the references given in the text, and to the www pages of the Consortium Project "Seismic Waves in 3-D Complex Structures"; see *http://seis.karlov.mff.cuni.cz/consort/main.htm* for some examples.

The whole book has a tutorial character. The equations presented are (in most cases) derived and discussed in detail. For this reason, the book is rather long. Owing to the extensive use of various matrix notations and to the applications of several coordinate systems and transformation matrices, the resulting equations are very concise and simply understandable from a seismological point of view. Although the equations are given in a concise and compact form, the whole book is written in an algorithmic way: most of the expressions are specified to the last detail and may be directly used for programming.

To write the complicated equations of this book in the most concise form, we use mostly matrix notation. To distinguish between 2×2 and 3×3 matrices, we shall use the circumflex ($\hat{}$) above the letter for 3×3 matrices. If the same letter is used for both 2×2 and 3×3 matrices; for example, $\mathbf{M}$ and $\hat{\mathbf{M}}$, matrix $\mathbf{M}$ denotes the 2×2 left upper submatrix of $\hat{\mathbf{M}}$:

$$\hat{\mathbf{M}} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}, \qquad \mathbf{M} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}.$$

Similarly, we denote by $\hat{\mathbf{q}} = (q_1, q_2, q_3)^T$ the 3×1 column matrix and by $\mathbf{q} = (q_1, q_2)^T$ the 2×1 column matrix. The symbol T as a superscript denotes the matrix transpose. Similarly, the symbol $-1T$ as a superscript denotes the transpose of the inverse, $\mathbf{A}^{-1T} = (\mathbf{A}^{-1})^T$, $\hat{\mathbf{A}}^{-1T} = (\hat{\mathbf{A}}^{-1})^T$.

In several places, we also use 4×4 and 6×6 matrices. We denote them by boldface letters in the same way as the 2×2 matrices; this notation cannot cause any misunderstanding.

In parallel with matrix notation, we also use component notation where suitable. The indices always have the form of right-hand suffixes. The uppercase suffixes take the values

1 and 2, lowercase indices 1, 2 and 3, and greek lowercase indices 1, 2, 3 and 4. In this way, M_{IJ} denote elements of $\mathbf{M}$ and M_{ij} elements of $\hat{\mathbf{M}}$. We also denote $f(x_i) = f(x_1, x_2, x_3)$, $f(x_I) = f(x_1, x_2)$, $[f(x_i)]_{x_k=0} = f(0, 0, 0)$, $[f(x_i)]_{x_K=0} = f(0, 0, x_3)$, $[f(x_i)]_{x_1=0} = f(0, x_2, x_3)$. The Einstein summation convention is used throughout the book. Thus, $M_{IJ}q_J = M_{I1}q_1 + M_{I2}q_2$ ($I = 1$ or 2), $M_{ij}q_j = M_{i1}q_1 + M_{i2}q_2 + M_{i3}q_3$ ($i = 1, 2$ or 3). Similarly, M_{iJ} denotes the elements of the 3×2 submatrix of matrix $\hat{\mathbf{M}}$.

We also use the commonly accepted notation for partial derivatives with respect to Cartesian coordinates x_i (for example, $\lambda_{,i} = \partial\lambda/\partial x_i$, $u_{i,ji} = \partial^2 u_i/\partial x_j \partial x_i$, $\sigma_{ij,j} = \partial\sigma_{ij}/\partial x_j$). In the case of velocities, we shall use a similar notation to denote the partial derivatives with respect to the ray-centered coordinates. For a more detailed explanation, see the individual chapters.

In some equations, the classical vector notation is very useful. We use arrows above letters to denote the 3-D vectors. In this way, any 3-D vector may be denoted equivalently as a 3×1 column matrix or as a vectorial form.

In complex-valued quantities, $z = x + iy$, the asterisk is used as a superscript to denote a complex-conjugate quantity, $z^* = x - iy$. The asterisk between two time-dependent functions, $f_1(t) * f_2(t)$, denotes the time convolution of these two functions, $f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t - \tau) d\tau$.

The book does not give a systematic bibliography on the seismic ray method. For many other references, see the books and review papers on the seismic ray method and on some related subjects (Červený and Ravindra 1971; Červený, Molotkov, and Pšenčík 1977; Hubral and Krey 1980; Hanyga, Lenartowicz, and Pajchel 1984; Bullen and Bolt 1985; Červený 1985a, 1985b, 1987a, 1989a; Chapman 1985, in press; Virieux 1996; Dahlen and Tromp 1998). The ray method has been also widely used in other branches of physics, mainly in electromagnetic theory (see, for example, Synge 1954; Kline and Kay 1965; Babich and Buldyrev 1972; Felsen and Marcuvitz 1973; Kravtsov and Orlov 1980).

CHAPTER TWO

The Elastodynamic Equation and Its Simple Solutions

The seismic ray method is based on asymptotic high-frequency solutions of the elastodynamic equation. We assume that the reader is acquainted with linear elastodynamics and with the simple solutions of the elastodynamic equation in a homogeneous medium. For the reader's convenience, we shall briefly discuss all these topics in this chapter, particularly the plane-wave and point-source solutions of the elastodynamic equation. We shall introduce the terminology, notations, and all equations we shall need in the following chapters. In certain cases, we shall only summarize the equations without deriving them, mainly if such equations are known from generally available textbooks. This applies, for example, to the basic concepts of linear elastodynamics. In other cases, we shall present the main ideas of the solution, or even the complete derivation. This applies, for example, to the Green functions for acoustic, elastic isotropic and elastic anisotropic homogeneous media.

In addition to elastic waves in solid isotropic and anisotropic models, we shall also study pressure waves in fluid models. In this case, we shall speak of the *acoustic case*. There are two main reasons for studying the acoustic case. The first reason is tutorial. All the derivations for the acoustic case are very simple, clear, and comprehensible. In elastic media, the derivations are also simple in principle, but they are usually more cumbersome. Consequently, we shall mostly start the derivations with the acoustic case, and only then shall we discuss the elastic case. The second reason is more practical. Pressure waves in fluid models are often used as a simple approximation of P elastic waves in solid models. For example, this approximation is very common in seismic exploration for oil.

The knowledge of plane-wave solutions of the elastodynamic equation in homogeneous media is very useful in deriving approximate high-frequency solutions of elastodynamic equation in smoothly inhomogeneous media. Such approximate high-frequency solutions in smoothly inhomogeneous media are derived in Section 2.4. In the terminology of the ray-series method, such solutions represent the zeroth-order approximation of the ray method. The approach we shall use in Section 2.4 is very simple and is quite sufficient to derive all the basic equations of the zeroth-order approximation of the ray method for acoustic, elastic isotropic, and elastic anisotropic structures. In the acoustic case, the approach yields the eikonal equation for travel times and the transport equation for scalar amplitudes. In the elastic case, it yields an approximate high-frequency decomposition of the wave field into the separate waves (P and S waves in isotropic; qP, qS1, and qS2 in anisotropic media). Thereafter, it yields the eikonal equations for travel times, the transport equations for amplitudes, and the rules for the polarization of separate waves.

Note that Section 2.4 deals only with the zeroth-order approximation of the ray-series method. The higher-order terms of the ray series are not discussed here, but will be considered in Chapter 5 (Sections 5.6 for the acoustic case and Section 5.7 for the elastic case). From the systematic and theoretical points of view, it would be more convenient to start the whole treatment directly with the ray-series method, and only then discuss the zeroth-order approximation, as a leading term of the ray-series method. The reason why we have moved the ray-series treatment to Sections 5.6 and 5.7 is again tutorial. The complete treatment of rays, ray-theory travel times, and paraxial methods in Chapters 3 and 4 is based on eikonal equations only. Similarly, all the treatments of ray amplitudes in Sections 5.1 through 5.5 are based on transport equations only. Thus, we do not need to know the higher-order terms of the ray series in Chapters 3 and 4 and in Sections 5.1 through 5.5; the results of Section 2.4 are sufficient there. Consequently, the whole ray-series treatment, which is more cumbersome than the derivation of Section 2.4, can be moved to Sections 5.6 and 5.7. Most of the recent applications of the seismic ray method are based on the zeroth-order approximation of the ray series. Consequently, most readers will be interested in the relevant practical applications of the seismic ray method, such as ray tracing, travel time, and ray amplitude computations. These readers need not bother with the details of the ray-series method; the zeroth-order approximation, derived in Section 2.4, is sufficient for them. The readers who wish to know more about the ray-series method and higher order terms of the ray series can read Sections 5.6 and 5.7 immediately after reading Section 2.4. Otherwise, no results of Sections 5.6 and 5.7 are needed in the previous sections.

Section 2.5 discusses the point-source solutions and appropriate Green functions for homogeneous fluid, elastic isotropic, and elastic anisotropic media. In all three cases, exact expressions for the Green function are derived uniformly. For elastic anisotropic media, exact expressions are obtained only in an integral form. Suitable asymptotic high-frequency expressions are, however, given in all three cases. These expressions are used in Chapter 5 to derive the asymptotic high-frequency expressions for the ray-theory Green function corresponding to an arbitrary elementary wave propagating in a 3-D laterally varying layered and blocked structure (fluid, elastic isotropic, elastic anisotropic).

The Green function corresponds to a point source, but it may be used in the representation theorem to construct considerably more complex solutions of the elastodynamic equation. If we are interested in high-frequency solutions, the ray-theory Green function may be used in the representation theorems. For this reason, representation theorems and the ray-theory Green functions play an important role even in the seismic ray method. The representation theorems are derived and briefly discussed in Section 2.6. The same section also discusses the scattering integrals and the first-order Born approximation. These integrals contain the Green function. If we use the ray-theory Green function in these integrals, the resulting scattering integrals can be used broadly in the seismic ray method and in relevant applications. Such approaches have recently found widespread applications in seismology and seismic exploration.

2.1 Linear Elastodynamics

The basic concepts and equations of linear elastodynamics have been explained in many textbooks and papers, including some seismological literature. We refer the reader to Bullen (1965), Auld (1973), Pilant (1979), Aki and Richards (1980), Hudson (1980a), Ben-Menahem and Singh (1981), Mura (1982), Bullen and Bolt (1985), and Davis (1988), where many other references can be found. For a more detailed treatment, see Love (1944), Landau and Lifschitz (1965), Fung (1965), and Achenbach (1975). Here we shall introduce only the most useful terminology and certain important equations that we shall need later. We shall mostly follow and use the notations of Aki and Richards (1980).

To write the equations of linear elastodynamics, some knowledge of tensor calculus is required. Because we wish to make the treatment as simple as possible, we shall use Cartesian coordinates x_i and Cartesian tensors only.

We shall use the Lagrangian description of motion in an elastic continuum. In the Lagrangian description, we study the motion of a particle specified by its original position at some reference time. Assume that the particle is located at the position described by Cartesian coordinates x_i at the reference time. The vector distance of a particle at time t from position $\vec{x}$ at the reference time is called the *displacement vector* and is denoted by $\vec{u}$. Obviously, $\vec{u} = \vec{u}(\vec{x}, t)$.

We denote the Cartesian components of the *stress tensor* by $\tau_{ij}(\vec{x}, t)$ and the Cartesian components of the *strain tensor* by $e_{ij}(\vec{x}, t)$. Both tensors are considered to be symmetric,

$$\tau_{ij} = \tau_{ji}, \qquad e_{ij} = e_{ji}. \tag{2.1.1}$$

The strain tensor can be expressed in terms of the displacement vector as follows:

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}). \tag{2.1.2}$$

The stress tensor $\tau_{ij}(\vec{x}, t)$ fully describes the stress conditions at any point $\vec{x}$. It can be used to compute *traction* $\vec{T}$ acting across a surface element of arbitrary orientation at $\vec{x}$,

$$T_i = \tau_{ij} n_j, \tag{2.1.3}$$

where $\vec{n}$ is the unit normal to the surface element under consideration.

The *elastodynamic equation* relates the spatial variations of the stress tensor with the time variations of the displacement vector,

$$\tau_{ij,j} + f_i = \rho \ddot{u}_i, \qquad i = 1, 2, 3. \tag{2.1.4}$$

Here f_i denote the Cartesian components of body forces (force per volume), and ρ is the density. The term with f_i in elastodynamic equation (2.1.4) will also be referred to as *the source term*. Quantities $\ddot{u}_i = \partial^2 u_i/\partial t^2$, $i = 1, 2, 3$, represent the second partial derivatives of u_i with respect to time (that is, the Cartesian components of *particle acceleration* $\ddot{\vec{u}}$). In a similar way, we shall also denote the Cartesian components of *particle velocity* $\partial u_i/\partial t$ by v_i or $\dot{u}_i$.

The introduced quantities are measured in the following units: stress τ_{ij} and traction T_i in pascals (Pa; that is, in kg m^{-1} s^{-2}), the components of body forces f_i in newtons per cubic meter (N/m^3; that is, in kg m^{-2} s^{-2}), density ρ in kilograms per cubic meter (kg m^{-3}), and displacement components u_i in meters (m). Finally, strain components e_{ij} are dimensionless.

2.1.1 Stress-Strain Relations

In a linear, anisotropic, perfectly elastic solid, the constitutive stress-strain relation is given by the *generalized Hooke's law*,

$$\tau_{ij} = c_{ijkl} e_{kl}. \tag{2.1.5}$$

Here c_{ijkl} are components of the elastic tensor. The elastic tensor has, in general, $3 \times 3 \times 3 \times 3 = 81$ components. These components, however, satisfy the following *symmetry relations*:

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}, \tag{2.1.6}$$

which reduce the number of independent components of the elastic tensor from 81 to 21.

The components c_{ijkl} of the elastic tensor are also often called *elastic constants, elastic moduli, elastic parameters, or stiffnesses*. In this book, we shall mostly call them elastic moduli. They are measured in the same units as the stress components (that is, in Pa = kg m^{-1} s^{-2}).

If we express e_{kl} in terms of the displacement vector components, see Equation (2.1.2), and take into account symmetry relations (2.1.6), we can also express Equation (2.1.5) in the following form:

$$\tau_{ij} = c_{ijkl} u_{k,l}. \tag{2.1.7}$$

The components of elastic tensor c_{ijkl} are also often expressed in an abbreviated Voigt form, with two indices instead of four. We shall denote these components by capital letters C_{mn}. C_{mn} is formed from c_{ijkl} in the following way: m corresponds to the first pair of indices, i, j and n to the second pair, k, l. The correspondence $m \to i, j$ and $n \to k, l$ is as follows: $1 \to 1, 1$; $2 \to 2, 2$; $3 \to 3, 3$; $4 \to 2, 3$; $5 \to 1, 3$; $6 \to 1, 2$.

Due to symmetry relations (2.1.6), the 6×6 matrix C_{mn} fully describes the elastic moduli of an arbitrary anisotropic elastic medium. It is also symmetric, $C_{mn} = C_{nm}$ and is commonly expressed in the form of a table containing 21 independent elastic moduli:

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{pmatrix}. \tag{2.1.8}$$

The elastic moduli C_{mn} below the diagonal $(m > n)$ are not shown because the table is symmetrical, $C_{mn} = C_{nm}$. The diagonal elements in the table are always positive for a solid medium, but the off-diagonal elements may be arbitrary (positive, zero, negative). Note that C_{mn} is not a tensor.

A whole hierarchy of various anisotropic symmetry systems exist. They are described and discussed in many books and papers; see, for example, Fedorov (1968), Musgrave (1970), Auld (1973), Crampin and Kirkwood (1981), Crampin (1989), and Helbig (1994). The most general is the *triclinic symmetry*, which may have up to 21 independent elastic moduli. In simpler (higher symmetry) anisotropic systems, the elastic moduli are invariant to rotation about a specific axis by angle $2\pi/n$ (n-fold axis of symmetry). We shall briefly discuss only two such simpler systems which play an important role in recent seismology and seismic exploration: orthorhombic and hexagonal.

In the *orthorhombic symmetry system*, three mutually perpendicular twofold axes of symmetry exist. The number of significant elastic moduli in the orthorhombic system is reduced to nine. If the Cartesian coordinate system being considered is such that its axes