This graduate/research-level text describes in a unified fashion the statistical mechanics of random walks, random surfaces and random higher-dimensional manifolds with an emphasis on the geometrical aspects of the theory and applications to the quantization of strings, gravity and topological field theory.

With chapters on random walks, random surfaces, two- and higher-dimensional quantum gravity, topological quantum field theories and Monte Carlo simulations of random geometries, the text provides a self-contained account of quantum geometry from a statistical field theory point of view. The approach uses discrete approximations and develops analytical and numerical tools. Continuum physics is recovered through scaling limits at phase transition points and the relation to conformal quantum field theories coupled to quantum gravity is described. The most important numerical work is covered, although the main aim is to develop mathematically precise results that have wide applications. Many diagrams and references are included.

This book will be of interest to graduate students and researchers in theoretical and statistical physics, and in mathematics.

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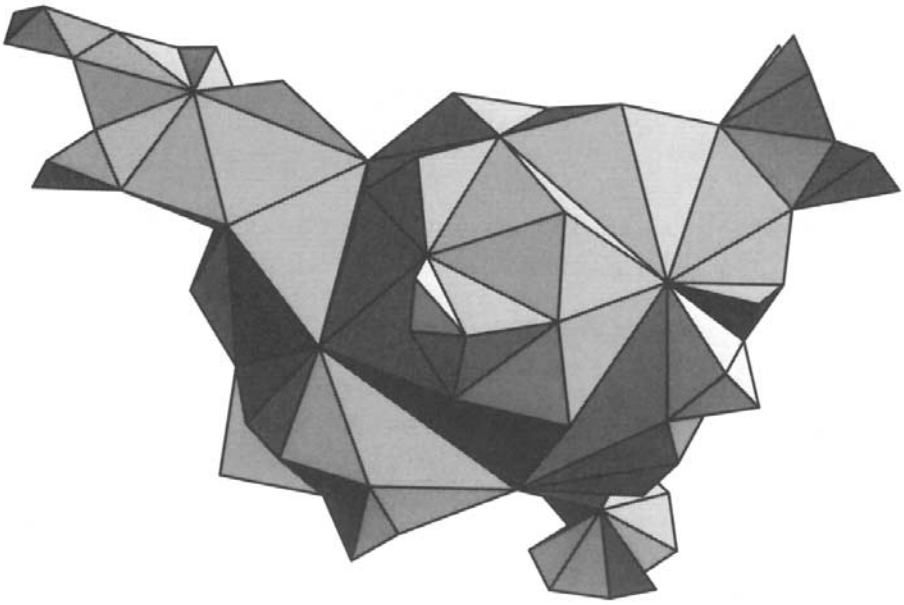

A generic universe in two-dimensional gravity, generated by the computer algorithms described in Chapter 5, and embedded in three-dimensional space for illustrative purposes.

Quantum Geometry

A statistical field theory approach

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Contents

Preface

The main topic of this book is the work that has been carried out during the last 15 years under the general heading of *random surfaces*. The original motivation for the study of random surfaces came from lattice gauge theory, where one can represent various quantities of interest as weighted sums over surfaces embedded in a hypercubic lattice. A few years later, with the resurrection of string theory, random surfaces were used as regularization of that theory and, most recently, random surface models have been applied to two-dimensional quantum gravity. There is also an impressive body of work on random surfaces that has been carried out by membrane physicists, as well as by condensed matter physicists, so one often finds mathematically identical problems being studied in different branches of physics. Random surfaces are therefore not a physical theory but, rather, a theoretical tool and a methodology that can be applied to various physical problems in the same way as random walks find applications in many branches of science. The formalism that has been developed to deal with random surfaces carries over to the study of higher-dimensional manifolds, which are important for quantizing gravity in higher dimensions.

We address this book primarily to advanced graduate students in theoretical physics but we hope that more experienced researchers in the field, as well as mathematicians, may find it useful. As far as applications are concerned, our point of view is very much biased towards string theory and quantum gravity, where the geometric point of view is most important and powerful. This perhaps justifies the book's title.

The purpose of writing this book was not to provide an encyclopaedic account of all the work on random geometry; rather, we have chosen to focus on mathematically precise results that have wide applications, as well as on key results from numerical simulations which have played an important role in guiding the development of the theory. We do not

provide complete proofs of the statements in the text where their content is not illuminating or where they are long and tedious, but try to warn the reader when the discussion becomes conjectural.

At the end of most chapters are notes that are intended to guide the reader to the original literature, as well as review articles and extensions of the discussion in the main text. There is no doubt that we have left out many important papers and we apologize in advance for all those omissions.

Important parts of the work reported in this book have been carried out in collaboration with our friends and colleagues A. Beliakova, D. Boulatov, J. Burda, M. Carfora, L. Chekhov, J. Fröhlich, J. Greensite, H. P. Jakobsen, J. Jurkiewicz, V. A. Kazakov, C. F. Kristjansen, Y. M. Makeenko, A. Marzuoli, R. Nest, P. Orland, B. Petersson, G. Thorleifsson, S. Varsted, Y. Watabiki and J. Wheater. We are indebted to J. Jurkiewicz, C. F. Kristjansen and Y. M. Makeenko for reading parts of the first draft of the book and pointing out many things that could be improved. M. Lund and G. Xander gave very valuable assistance with the diagrams, the references and the index.

Jan Ambjørn
Bergfinnur Durhuus
Thordur Jonsson

Notation

The purpose of this note is to explain a few notational conventions that we have used throughout the text, some of which may not be completely standard.

If f is a function of a positive real variable x, then the equation

$$f(x) = O(x^n)$$

means that there is a constant $C > 0$ such that

$$f(x) \leq Cx^n$$

for all values of x or for x in an asymptotic region, i.e. for either large or small x, depending on the context.

If f is as above, then the equation

$$f(x) = o(x^n)$$

means that

$$\lim_{x \to 0} \frac{f(x)}{x^n} = 0.$$

We use $O(x, y)$ as shorthand notation for $O(x) + O(y)$, and similarly for $o(x, y)$.

Finally, let f and g be functions of a real or complex variable x. We employ the notation

$$f(x) \sim g(x)$$

to mean one of three things: which one we have in mind is either explained explicitly in the text or it is supposed to be clear from the context. The first meaning is that the functions f and g are asymptotic as x approaches some limiting value x_0, i.e.

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = C,$$

where C is a non-zero constant. It should always be clear from the context what x_0 is. In particular, x_0 can be infinite. The second meaning is that

$$\lim_{x \to x_0} \frac{\log f(x)}{\log g(x)} = C.$$

This case arises in particular when we wish to say that two functions have the same exponential decay. The third meaning is that f and g have a singularity of the same kind at the limiting point x_0. For example, if f and g are real analytic for $x > x_0$ and the nth derivative of f, $f^{(n)}$, is the lowest derivative of f which does not have a finite limit as $x \downarrow x_0$, then $f(x) \sim g(x)$ means that the limit

$$\lim_{x \to x_0} \frac{f^{(n)}(x)}{g^{(n)}(x)}$$

exists and is non-zero.