Finite-element plasticity and metalforming analysis
Finite-element plasticity and metalforming analysis

G. W. Rowe
Professor of Mechanical Engineering
University of Birmingham

C. E. N. Sturgess
Jaguar Professor of Automotive Engineering
University of Birmingham

P. Hartley
Senior Lecturer, Department of Manufacturing Engineering
University of Birmingham

I. Pillinger
Senior Computer Officer, Department of Mechanical Engineering
University of Birmingham

CAMBRIDGE UNIVERSITY PRESS
Cambridge
New York Port Chester
Melbourne Sydney
Contents

Preface xi
Acknowledgements xiii
Nomenclature xv

1 General introduction to the finite-element method 1
  1.1 Introduction 1
  1.2 Earlier theoretical methods for metalforming analysis 3
  1.3 Basic finite-element approach 4
  1.4 General procedure for structural finite-element analysis 5
  1.5 Simple application of elastic FE analysis: a tensile test bar 7
    1.5.1 Discretisation 7
    1.5.2 Interpolation 7
    1.5.3 Stiffness matrices of the elements 7
    1.5.4 Assembly of element stiffness matrices 10
    1.5.5 Boundary conditions 10
    1.5.6 Numerical solution for the displacements 10
    1.5.7 Strains and stresses in the elements 12
    References 12

2 Basic formulation for elastic deformation 14
  2.1 Types of elements 14
    2.1.1 Linear elements 15
    2.1.2 Plane-strain triangular elements 16
    2.1.3 Linear quadrilateral elements 16
    2.1.4 Higher-order (quadratic) elements 16
    2.1.5 Three-dimensional elements 17
    2.1.6 Size of elements 17
    2.1.7 Shape and configuration of elements: aspect ratio 18
  2.2 Continuity and equilibrium 18
  2.3 Interpolation functions 19
<table>
<thead>
<tr>
<th>vi</th>
<th>Contents</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.3.1</td>
<td>Polynomials</td>
</tr>
<tr>
<td>2.3.2</td>
<td>Convergence</td>
</tr>
<tr>
<td>2.4</td>
<td>Displacement vector ( \mathbf{u} ) and matrix ([B] )</td>
</tr>
<tr>
<td>2.5</td>
<td>The ([D] ) matrix of elastic constants</td>
</tr>
<tr>
<td>2.5.1</td>
<td>General expression</td>
</tr>
<tr>
<td>2.5.2</td>
<td>Plane elastic stress</td>
</tr>
<tr>
<td>2.5.3</td>
<td>Plane elastic strain</td>
</tr>
<tr>
<td>2.5.4</td>
<td>Axial symmetry</td>
</tr>
<tr>
<td>2.6</td>
<td>Formulation of the element stiffness matrix ([K]_e )</td>
</tr>
<tr>
<td>2.7</td>
<td>Formulation of the global stiffness matrix ([K] )</td>
</tr>
<tr>
<td>2.7.1</td>
<td>Assembly of element matrices</td>
</tr>
<tr>
<td>2.7.2</td>
<td>Properties of ([K] )</td>
</tr>
<tr>
<td>2.8</td>
<td>General solution methods for the matrix equations</td>
</tr>
<tr>
<td>2.8.1</td>
<td>Direct solution methods</td>
</tr>
<tr>
<td>2.8.2</td>
<td>Indirect solution methods</td>
</tr>
<tr>
<td>2.8.3</td>
<td>Iteration to improve a direct solution</td>
</tr>
<tr>
<td>2.9</td>
<td>Boundary conditions</td>
</tr>
<tr>
<td>2.10</td>
<td>Variational methods</td>
</tr>
<tr>
<td>2.10.1</td>
<td>Variational method of solution in continuum theory</td>
</tr>
<tr>
<td>2.10.2</td>
<td>Approximate solutions by the Rayleigh–Ritz method</td>
</tr>
<tr>
<td>2.10.3</td>
<td>Weighted residuals</td>
</tr>
<tr>
<td>2.10.4</td>
<td>Galerkin’s method</td>
</tr>
<tr>
<td>2.11</td>
<td>The finite-difference method</td>
</tr>
<tr>
<td>References</td>
<td>39</td>
</tr>
</tbody>
</table>

3 Small-deformation elastic-plastic analysis | 40 |
3.1 Introduction | 40 |
3.2 Elements of plasticity theory | 40 |
3.2.1 Yielding | 40 |
3.2.2 Deviatoric and generalised stresses | 42 |
3.2.3 Constancy of volume in plastic deformation | 42 |
3.2.4 Decomposition of incremental strain | 43 |
3.2.5 Generalised plastic strain | 43 |
3.2.6 The relationship of stress to strain increment: the Prandtl–Reuss equations | 44 |
3.2.7 Elastic-plastic constitutive relationship | 45 |
3.2.8 Strain hardening in FE solutions | 45 |
3.2.9 Force/displacement relationship: the stiffness matrix \([K] \) | 46 |
3.3 Example analysis using the small-deformation formulation | 46 |
References | 48 |

4 Finite-element plasticity on microcomputers | 49 |
4.1 Microcomputers in engineering | 49 |
4.2 Non-linear plasticity demonstration programs on a microcomputer using BASIC | 50 |
<table>
<thead>
<tr>
<th>Contents</th>
<th>vii</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.2.1 Introduction</td>
<td>50</td>
</tr>
<tr>
<td>4.2.2 Structure of the finite-element program in BASIC</td>
<td>51</td>
</tr>
<tr>
<td>4.2.3 Simple application and comparison to mainframe results</td>
<td>52</td>
</tr>
<tr>
<td>4.3 A 'large' FORTRAN-based system for non-linear finite-element plasticity</td>
<td>56</td>
</tr>
<tr>
<td>4.3.1 Hardware selection</td>
<td>56</td>
</tr>
<tr>
<td>4.3.2 Program transfer</td>
<td>58</td>
</tr>
<tr>
<td>4.3.3 Applications of the non-linear system</td>
<td>58</td>
</tr>
<tr>
<td>4.3.3.1 Axial symmetric upsetting</td>
<td>58</td>
</tr>
<tr>
<td>4.3.3.2 Cold heading</td>
<td>60</td>
</tr>
<tr>
<td>4.3.4 Overcoming the FORTRAN compiler limitations</td>
<td>61</td>
</tr>
<tr>
<td>4.3.5 Introducing an improved FORTRAN compiler and the 8087 maths co-processor</td>
<td>62</td>
</tr>
<tr>
<td>4.3.5.1 System improvements</td>
<td>62</td>
</tr>
<tr>
<td>4.3.5.2 Analysis of upsetting and heading with refined meshes</td>
<td>63</td>
</tr>
<tr>
<td>4.4 Summary</td>
<td>64</td>
</tr>
<tr>
<td>References</td>
<td>65</td>
</tr>
</tbody>
</table>

## 5 Finite-strain formulation for metalforming analysis

5.1 Introduction 66

5.2 Governing rate equations 67

5.2.1 Rate of potential energy for updated-Lagrangian increment 67

5.2.2 Minimisation of rate of potential energy 68

5.2.3 Incorporation of strain rate 68

5.2.4 Importance of correct choice of stress rate 69

5.3 Governing incremental equations 70

5.3.1 Modification of rate expression 70

5.3.2 Effect of rotation 71

5.3.3 LCR expression for strain increment 72

5.4 Elastic-plastic formulation 73

5.4.1 Yield criterion 73

5.4.2 Elastic-plastic flow rule 74

5.4.3 Elastic-plastic constitutive relationship 74

5.4.4 Effect of plastic incompressibility 74

5.4.5 Element-dilatation technique 75

5.5 Element expressions 77

5.5.1 Interpolation of nodal displacement 77

5.5.2 Incremental element-stiffness equations 78

References 78

## 6 Implementation of the finite-strain formulation

6.1 Introduction 80

6.2 Performing an FE analysis – an overview 80

6.3 Pre-processing 82

6.3.1 Mesh generation 82
## Contents

6.3.2 Boundary conditions  83
6.3.3 Material properties  85
6.3.4 Deformation  85
6.3.5 Initial state parameters  87
6.4 FE Calculation  87
  6.4.1 Input of data  87
  6.4.2 Displacement of surface nodes  88
  6.4.3 Assembly and solution of the stiffness equations  90
  6.4.3.1 Incorporation of frictional restraint  90
  6.4.3.2 Solution techniques  92
  6.4.3.3 Yield-transition increments  95
  6.4.4 Updating of workpiece parameters  95
  6.4.4.1 Strain components and rotational values  95
  6.4.4.2 Deviatoric stress  96
  6.4.4.3 Hydrostatic stress  99
  6.4.4.4 External forces  100
  6.4.5 Strain rate  100
  6.4.4.6 Temperature  101
  6.4.5 Output of results  107
  6.5 Post-processing  108
  6.6 Special techniques  108
  6.6.1 Processes involving severe deformation  108
  6.6.2 Steady-state processes  111
  6.6.3 Three-dimensional analyses  112
  References  114

7 Practical applications  116
  7.1 Introduction  116
  7.2 Forging  117
    7.2.1 Simple upsetting  117
    7.2.2 Upset forging  122
    7.2.3 Heading  122
    7.2.4 Plane-strain side-pressing  125
    7.2.5 Rim-disc forging  131
    7.2.6 Extrusion-forging  135
    7.2.7 ‘H’-section forging  137
    7.2.8 Forging of a connecting rod  145
  7.3 Extrusion  149
    7.3.1 Forward extrusion  149
  7.4 Rolling  153
    7.4.1 Strip rolling  153
    7.4.2 Slab rolling  155
  7.5 Multi-stage processes – forging sequence design  159
    7.5.1 Automobile spigot  159
    7.5.2 Short hollow tube (gudgeon pin)  163
    References  167
<table>
<thead>
<tr>
<th>Contents</th>
<th>ix</th>
</tr>
</thead>
<tbody>
<tr>
<td>8 Future developments</td>
<td>169</td>
</tr>
<tr>
<td>8.1 Introduction</td>
<td>169</td>
</tr>
<tr>
<td>8.2 Developments in software</td>
<td>170</td>
</tr>
<tr>
<td>8.3 Advances in modelling</td>
<td>170</td>
</tr>
<tr>
<td>8.4 Material properties</td>
<td>172</td>
</tr>
<tr>
<td>8.5 Post-processing</td>
<td>173</td>
</tr>
<tr>
<td>8.6 Expert systems</td>
<td>174</td>
</tr>
<tr>
<td>8.7 Hardware</td>
<td>174</td>
</tr>
<tr>
<td>8.8 Applications in the future</td>
<td>175</td>
</tr>
<tr>
<td>References</td>
<td>176</td>
</tr>
</tbody>
</table>

**Appendices**

<table>
<thead>
<tr>
<th>Contents</th>
<th>l78</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Derivation of small-strain $[B]$ matrix for 2-D triangular element</td>
<td>178</td>
</tr>
<tr>
<td>2 Derivation of elastic $[D]$ matrix</td>
<td>181</td>
</tr>
<tr>
<td>3 Derivation of elastic-plastic $[D]$ matrix</td>
<td>184</td>
</tr>
<tr>
<td>4 Derivation of small-strain stiffness matrix $[K]$ for plane-stress triangular element</td>
<td>188</td>
</tr>
<tr>
<td>5 Solution of stiffness equations by Gaussian elimination and back-substitution</td>
<td>191</td>
</tr>
<tr>
<td>6 Imposition of boundary conditions</td>
<td>196</td>
</tr>
<tr>
<td>7 Relationship between elastic moduli $E$, $G$ and $\kappa$</td>
<td>202</td>
</tr>
<tr>
<td>8 Vectors and tensors</td>
<td>205</td>
</tr>
<tr>
<td>9 Stress in a deforming body</td>
<td>209</td>
</tr>
<tr>
<td>10 Stress rates</td>
<td>216</td>
</tr>
<tr>
<td>11 Listing of BASIC program for small-deformation elastic-plastic FE analysis</td>
<td>218</td>
</tr>
</tbody>
</table>

**Bibliography**

<table>
<thead>
<tr>
<th>Contents</th>
<th>246</th>
</tr>
</thead>
</table>

**Index**

| Contents     | 294 |
Preface

More and more sectors of the metalforming industry are beginning to recognise the benefits that finite-element analysis of metal-deformation processes can bring in reducing the lead time and development costs associated with the manufacture of new components. Finite-element analyses of non-linear problems, such as metal deformation, require powerful computing facilities and large amounts of computer running time but advances in computer technology and the falling price of hardware are bringing these techniques within the reach of even the most modest R & D department.

Finite-element programs specially designed for the analysis of metalforming processes (such as the program EPFEP 3, which is used for the majority of examples in this monograph) are now commercially available. As a result, there is an urgent need for a text that explains the principles underlying finite-element metalforming analysis to the people who are starting to use these techniques industrially, and to those undertaking the undergraduate and postgraduate engineering courses in the subject that industry is beginning to demand. One of the aims of this monograph is to fulfil that need.

The first chapters provide an introduction to the application of finite-element analysis to metalforming problems, starting from the basic ideas of the finite-element method, and developing these ideas firstly to the study of linear elastic deformation and then to the examination of non-linear elastic-plastic processes involving small amounts of deformation. No previous knowledge of finite elements is required, although the reader is assumed to be familiar with the use of matrices.

Chapter 4 describes a program, written in BASIC for a PC-type desktop computer, that uses the simple, small-deformation formulation. Although restricted to two-dimensional metal flow, low levels of deformation and a small

Copyright © 1987 University of Birmingham All Rights Reserved.
number of elements, this program demonstrates the use of finite-element analysis in metalforming and is particularly useful for tuition purposes. A full listing is given in Appendix 11. Chapter 4 also discusses implementations of the small-deformation theory on more powerful microcomputers that have larger memories and access to a FORTRAN compiler. These programs are capable of more detailed analysis, and can be used for the study of many simple two-dimensional processes.

The application of finite-element techniques to metalforming is still the subject of much research, and the second aim of this monograph is to present the current state of knowledge to those new to the field. Chapter 5 therefore examines in detail the theoretical aspects of a finite-element model of large-deformation plasticity, and Chapter 6 explains how this theory can be implemented in a practical program for the study of complex metalforming problems. Chapter 7 describes a wide variety of practical applications of the finite-element programs.

Although, where appropriate, we mention some of the other finite-element techniques that have been applied to plasticity problems, this monograph is written from the standpoint of an elastic-plastic approach to metal deformation. We are firmly of the opinion that the various simplifications that are sometimes made are a false economy and a serious hindrance to the further development of the finite-element study of metal flow.

The last chapter of this monograph identifies the main areas where we believe that future developments will take place, or at least those areas in which work needs to be done. Unfortunately, the gaps in the present state of knowledge of metalforming processes and the deficiencies in the available models of metal deformation are all too obvious. We hope that this monograph will provide a stimulus to others to attempt to remedy these failings.

G.W.R.
C.E.N.S.
P.H.
I.P.
Acknowledgements

Many people have been associated over the years with the Finite-element Plasticity Group at the University of Birmingham and have contributed, directly or indirectly, to the development of large-deformation elastic-plastic finite-element techniques. We are particularly indebted to Profs. C. Liu and K. Wang, Drs S.E. Clift, A.A.M. Hussin, A.A.K. Al-Senad, A.J. Eames and J. Salimi and Mr K. Kawamura, whose research work has influenced the contents of this monograph. We should also like to thank Mr S.K. Chanda, Profs. S. Ikeda and K. Kato, as well as the above persons, for providing the numerous examples of finite-element metalforming applications that are to be found in the text.

In addition, both undergraduate and postgraduate students have contributed through a variety of short projects. Many of the overseas research students and visiting research fellows have been able to work with us at Birmingham through the generosity of their respective governments and, in some cases, with additional help from the British Council. We are grateful to these bodies for their support.

Much of our research work has been carried out with the financial support of the UK Science and Engineering Research Council, and more recently the ACME Directorate, and with computer facilities provided by the Centre for Computing and Computer Science at the University of Birmingham and the University of Manchester Regional Computer Centre.

Our contacts with industry have been particularly useful to us and we are grateful for the advice and help we have obtained, especially from Alcan Plate, Austin-Rover, Automotive Products, GKN and High Duty Alloys.

We are very grateful to Mr B. Van Bael for checking the manuscript and particularly for wading through the mathematics. It is to his credit if the text emerges free of error. If any mistakes remain, they are entirely our fault, not his.

We were very saddened by the death of Professor Rowe during the preparation of this monograph. Much of the work described in the following pages was begun at his instigation and carried out under his guidance. Indeed, that the monograph
came into existence at all is very much due to his energy and his enthusiasm. We owe a great debt to him and feel privileged to have known him and to have worked with him over many years. In completing the book, we have been very conscious of the high standards he set in everything he wrote. We hope that the result does not compare too unfavourably with those standards. We should like to dedicate this work to him.

C. E. N. S
P. H.
I. P.
Nomenclature

Use of subscripts and superscripts

lower-case subscript in italic type:
usually indicates a Cartesian component of a quantity, e.g. $x_i$. If preceded by a comma, it indicates a derivative with respect to a particular Cartesian co-ordinate, e.g. $u_{,i} = \partial u / \partial x_i$.

lower-case subscript in bold type:
indicates the quantity associated with a particular element, e.g. $L_i$.

upper-case subscript in italic type:
indicates the quantity associated with a particular node of an element or body, e.g. $N_k$.

Greek subscript:
indicates the quantity associated with a particular degree of freedom of the whole workpiece. In the simple one-dimensional examples considered in the early part of this monograph, a Greek subscript therefore denotes the value of a quantity for a particular node of the workpiece, e.g. $d_s$.

lower-case superscript in italic type:
used in the notation for contravariant and mixed tensors, but only in Appendix 8 (vectors and tensors).

superscript in normal type:
is a label, rather than a numerical index, denoting a particular type of subset of a quantity, e.g. $\Delta T^\text{ref}$. Some frequently-used superscripts are:
- $T$ transpose of vector or matrix
- $p$ plastic portion
- $(\prime)$ for tensors – deviatoric component, e.g. $\sigma_\prime$
- for positional quantities – transformed position, e.g. $x'_i$
Nomenclature

for scalars – derivative, e.g. $Y''$

Other examples are given in the list of symbols below.

**lower-case superscript in parentheses:** indicates the value of a quantity for a particular iteration, e.g. $\Delta d^{(2)}$. The parentheses are used to avoid confusion with an exponent.

**Greek superscript:** indicates the quantity associated with a particular face of the element under consideration, e.g. $A^*$. Specific values of bold subscripts are printed using bold numerals but when numbers are substituted for italic subscripts normal type is used. No distinction is made between numbers substituted for lower-case subscripts and those substituted for Greek subscripts. The meaning is made clear by the context. The context will also make clear whether, for example, $X^i$ means the square of the value of $X$, or the second contravariant component of the vector $X$.

The different types of subscript and superscript may be used together, e.g. $f_{sa}$ or $q^{[ii]}$.

**Frequently-used qualifying symbols**

A dot above a quantity denotes the time derivative, e.g. $\dot{a}_{sa}$.

A bar above a tensor denotes the generalised value, e.g. $\bar{a}$.

**Representation of vectors and matrices**

A vector is represented by bold type, a matrix by a symbol enclosed in square brackets. Both may also be notated by specifying the symbol (without bold type or brackets) with algebraic subscripts (and possibly superscripts). This may represent a particular element of the vector or matrix, or may be intended to stand for the whole array, with the indices implicitly varying over all their possible values, e.g. $[K] = K_{ij}$. When the elements of a vector are written out explicitly, they are enclosed in parentheses. If they are written as a row vector, the elements are separated by commas. The elements of a matrix are simply enclosed in square brackets, without any commas.

**List of symbols**

- $A$ area
- $A_{1234}$ area of quadrilateral 1234
- $A_i$ area of element $i$
- $A^a$ area of face $a$ of element
- $a$ coefficient of general polynomial shape function. Also coefficient
of 1-D linear displacement function

\( a_i \) coefficient of linear function for displacement in \( x_i \) direction

\( a_{ij} \) coefficient of expression determining \( i \)th reference co-ordinate of a point as a quadratic function of local curvilinear co-ordinates \( X^\prime \)

\( da \) infinitesimal area at \( P \) at time \( t \)

\( da' \) infinitesimal area at \( P' \) at time \( t+\,dt \)

\( B \) coefficient of yield-stress function

\[ [B] \] matrix that may be used to express the strain at a point within an element in terms of the displacement of its nodes

\[ [B_i] \] strain/nodal displacement matrix for element \( i \)

\[ \tilde{B}_{ijmn} \] coefficient of LCR strain increment/nodal displacement increment

\( \tilde{B}_{ijmn} \) coefficient of defining matrix for LCR strain increment/nodal displacement increment

\( (= \frac{1}{2} (\tilde{B}_{ijmn} + \tilde{B}_{nijn}) ) \)

\( b \) coefficient of general polynomial shape function. Also coefficient of 1-D linear displacement function

\( b_i \) coefficient of general linear shape function

\( b_{ij} \) coefficient of linear function for displacement in \( x_i \) direction. Also coefficient of expression determining \( i \)th reference co-ordinate of a point as a quadratic function of local curvilinear co-ordinate \( X^\prime \)

\( C^+, C^- \) coefficients of expression determining rate of flow of heat at element centroid as a function of the difference in temperature between the centroid and the point on the \( j \)th local curvilinear axis with local co-ordinate +1 or -1

\( c \) coefficient of general polynomial shape function. Also thermal capacity per unit volume

\( c_i \) coefficient of linear combination of functions that form \( \hat{\phi} \)

\[ [D] \] 6 \times 6 stress/strain constitutive matrix such that \( \sigma = [D] \epsilon \)

\[ [D^e] \] elastic part of elastic-plastic constitutive matrix

\[ [D^p] \] elastic-plastic part of constitutive matrix

\[ [D_i] \] constitutive matrix for element \( i \)

\( D_{ij} \) entry in row \( i \), column \( j \) of \( [D] \)

\( D_{ijkl} \) coefficient of constitutive tensor relating Jaumann rate of stress \( \dot{\sigma}_{ij} \) to component \( \dot{\epsilon}_{ij} \) of strain-rate tensor

\( d \) displacement. Also thickness of friction layer

\( d \) global nodal displacement vector

\( d' \) approximate solution of equations derived by incomplete Choleski conjugate-gradient method (\( = [U']d \))

\( d_\alpha \) component \( \alpha \) of the nodal displacement vector. For 1-D problems, this is the displacement of node \( \alpha \)
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d'_n$</td>
<td>component $n$ of nodal displacement vector after modification during Gaussian elimination procedure</td>
</tr>
<tr>
<td>$d_i$</td>
<td>vector of components of displacement of node $i$</td>
</tr>
<tr>
<td>$d_i^i$</td>
<td>vector of components of displacement of nodes of element $i$</td>
</tr>
<tr>
<td>$d_i^f$</td>
<td>vector of components of displacement of node $i$ in rotated axis system</td>
</tr>
<tr>
<td>$d_{il}$</td>
<td>displacement of node $l$ of element in $x_i$ direction</td>
</tr>
<tr>
<td>$\Delta d$</td>
<td>representative displacement increment</td>
</tr>
<tr>
<td>$\Delta d^e$</td>
<td>first estimate of displacement increment during secant-modulus technique, evaluated by solving stiffness equations obtained for initial strain $\epsilon^i$ and stress $\sigma^i$</td>
</tr>
<tr>
<td>$\Delta d^f$</td>
<td>second estimate of displacement increment during secant-modulus technique, evaluated by solving stiffness equations obtained for mid-increment strain $\epsilon^m$ and stress $\sigma^m$</td>
</tr>
<tr>
<td>$\Delta d^{(i)}$</td>
<td>$i$th correction to the incremental displacement in initial-stiffness iteration</td>
</tr>
<tr>
<td>$\Delta d_{hl}$</td>
<td>change in $d_{hl}$ during an increment of deformation</td>
</tr>
<tr>
<td>$\Delta d$</td>
<td>error in estimate of $d$</td>
</tr>
<tr>
<td>$E$</td>
<td>Young's Modulus of elasticity</td>
</tr>
<tr>
<td>$E_i$</td>
<td>Young's Modulus for the material of element $i$</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>small extension of an element</td>
</tr>
<tr>
<td>$F$</td>
<td>functional of state functions in variational method. Also number of faces of an element in contact with a die</td>
</tr>
<tr>
<td>$f$</td>
<td>force</td>
</tr>
<tr>
<td>$f'$</td>
<td>force acting at a point $P$ of a body</td>
</tr>
<tr>
<td>$f'^i$</td>
<td>right-hand side of equations derived by incomplete Choleski conjugate-gradient method ($= [L^T]^Tf$)</td>
</tr>
<tr>
<td>$f^i$</td>
<td>force acting at time $t + \Delta t$ on an infinitesimal plane with area $da'$ and a normal that has reference components equal to $N_i^i$</td>
</tr>
<tr>
<td>$f^l$</td>
<td>global force vector corresponding to a global displacement vector of $d + \Delta d$</td>
</tr>
<tr>
<td>$f_a$</td>
<td>component $a$ of the nodal force vector. For 1-D problems, this is the resultant force at node $a$</td>
</tr>
<tr>
<td>$f_a$</td>
<td>component $a$ of the nodal force vector after modification during the Gaussian elimination procedure</td>
</tr>
<tr>
<td>$f_l^i$</td>
<td>contravariant reference component of $f_l$</td>
</tr>
<tr>
<td>$f_i$</td>
<td>vector of components of force applied to node $i$</td>
</tr>
<tr>
<td>$f_i$</td>
<td>vector of components of force applied to nodes of element $i$</td>
</tr>
<tr>
<td>$f_i^f$</td>
<td>vector of components of force applied to node $i$ in rotated axis system</td>
</tr>
</tbody>
</table>
\( f' \) contravariant reference component of \( f' \)
\( f_{s_i} \) component of force in \( x_i \) direction applied to node \( I \) of element
\( f_{s_a} \) force applied to element \( i \) at node \( a \)
\( \Delta f \) representative force increment
\( \Delta f^{(0)} \) resultant force calculated in \( i \)th iteration of initial-stiffness technique
\( \Delta f^{(0)} \) force in equilibrium with \( \Delta d^{(0)} \) in initial-stiffness iteration
\( \Delta f_s \) change in \( f_s \) during an increment of deformation
\( G \) Rigidity (shear) Modulus of elasticity \((= E/(1+\nu))\). Also functional of state functions used in variational method
\( G_i \) basis of 3-D vector space
\( G_i' \) basis of 3-D vector space (dual of \( G_i \))
\( g_i \) stiffness coefficient of element \( i \) \((= A_i E_i / L_i)\)
\( g_i(x) \) one of the simple functions of position used to define an approximation to the function of state in variational method
\( g_i' \) basis of 3-D vector space
\( g_i' \) basis of 3-D vector space (dual of \( g_i \))
\( H \) initial height of billet
\( h \) final height of billet
\( h(d) \) scalar function of \( n \) variables that has a minimum value for the vector solution of the stiffness equations
\( h_i \) potential energy used in the evaluation of rotational values \((= q_i q_i')\)
\( l \) potential energy; more generally, functional that needs to be minimised in order to determine the correct solution for the function of state in a physical system
\( l_i \) potential energy of element \( i \)
\( J \) Jacobian of transformation matrix \((= \text{det}(x'^i))\)
\( K \) stiffness
\( [K] \) global stiffness matrix
\( [K'] \) matrix derived by incomplete Choleski conjugate-gradient method \((=[L']^{-1} [K][U']^{-1})\)
\( [K_i] \) stiffness matrix of element \( i \)
\( K_{sp} \) entry in global stiffness matrix
\( K_{sp} \) entry in global stiffness matrix after modification during Gaussian elimination procedure
\( [K_{ij}] \) 2×2 (for 2-D) or 3×3 (for 3-D) submatrix of coefficients of global stiffness matrix relating components of force at node \( I \) to components of displacement at node \( J \)
\( K_{n,m} \) coefficient of element stiffness matrix relating component \( n \) of incremental displacement of node \( J \) to component \( m \) of incremental force applied to node \( I \)
Nomenclature

\( K_{peln} \) coefficient of element deformation stiffness matrix
\( K_{seln} \) coefficient of element stress-increment correction matrix
\( K_{deln} \) coefficient of element dilatation matrix
\( k \) shear yield stress. Also thermal conductivity
\( k^{+}, k^{-} \) coefficients of heat transfer between die and face of element with centre on positive or negative \( j \)th local curvilinear axis
\( L \) length
\( [L] \) lower-triangular matrix derived from \( [K] \) during Choleski decomposition
\( [L^+] \) approximation to \( [L] \) having a pattern of sparseness similar to that of \( [K] \)
\( L_{i} \) length of element \( i \)
\( P^{+}, P^{-} \) distance between centroid of element and face with centre on positive or negative \( j \)th local curvilinear axis
\( m \) friction factor. Also exponent in yield-stress function
\( m^{r} \) friction factor associated with die in contact with face \( e \) of element
\( \Delta m \) proportionality factor in modified Prandtl–Reuss equations
\( N \) number of nodes in an FE discretisation. Also number of steps in thermal calculation for an increment of deformation
\( N(x) \) general shape function of position
\( N_{i} \) covariant component of \( n \) in the convected co-ordinate system
\( N_{i}(x) \) shape function of position relating to the contribution from node \( l \)
\( N_{ij} \) gradient of \( N_{i}(x) \) \( (\equiv \partial N_{i}(x_{i})/\partial x_{i}) \)
\( n \) number of degrees of freedom in an FE discretisation
\( n^{r} \) normal to infinitesimal surface at \( P \) at time \( t \). Also normal to boundary surface at \( P \) at start of increment
\( n_{i} \) covariant components of \( n \) in reference co-ordinate system
\( n_{r} \) covariant components of \( n^{r} \) in reference co-ordinate system
\( O \) origin of reference co-ordinate system
\( O^{r} \) origin of convected co-ordinate system
\( P \) location of infinitesimal region of body at time \( t \). Also coefficient used in calculating number of thermal steps
\( P^{r} \) location of infinitesimal region of body at time \( t+\Delta t \)
\( p \) coefficient of general polynomial shape function. Also coefficient of quadratic function of \( \Delta m = (3\theta_{i}^{2}/2) \)
\( p(R) \) function of residual \( R \) in weighted residual method
\( p^{1}, p^{2}, p^{3} \) planes of constraint of nodes
\( p_{i}^{*} \) result of dividing \( h_{i} \) by \( i \)th estimate of \( q_{i} \)
\( q \) coefficient of quadratic function of \( \Delta m = (-3\theta_{i}^{2}(\sigma^{e}_{i}+\Delta\sigma^{e}_{i})) \)
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q^0 )</td>
<td>symmetric part of deformation matrix ( x^{ij} )</td>
</tr>
<tr>
<td>( q_0 )</td>
<td>symmetric part of deformation matrix ( x_0^{ij} )</td>
</tr>
<tr>
<td>( q_0^{i} )</td>
<td>( i )th estimate of ( q_0 ) using Newton's method of solving equation ( q_a q_0 = h_q )</td>
</tr>
<tr>
<td>( \delta Q^i )</td>
<td>increase in energy of element in one step of thermal calculation due to conduction of heat into and out of element</td>
</tr>
<tr>
<td>( \delta Q^0 )</td>
<td>increase in energy of element in one step of thermal calculation due to work of deformation</td>
</tr>
<tr>
<td>( \delta Q^d )</td>
<td>increase in energy of element in one step of thermal calculation due to frictional heating at boundaries</td>
</tr>
<tr>
<td>( R )</td>
<td>residual functional of state functions in weighted residual method</td>
</tr>
<tr>
<td>([R])</td>
<td>orthonormal (rotational) matrix relating global to rotated components of a vector</td>
</tr>
<tr>
<td>( r )</td>
<td>order of polynomial interpolation function. Also coefficient of quadratic function of ( \Delta m ) ( (= 3(\sigma_0^0 + \Delta \sigma_0^0)^2 / 2) )</td>
</tr>
<tr>
<td>( r_i )</td>
<td>right-hand side vector of equations obtained by Gauss-Jordan elimination</td>
</tr>
<tr>
<td>( r^i )</td>
<td>rotational (orthogonal) part of deformation matrix ( x^{ij} )</td>
</tr>
<tr>
<td>( r_0 )</td>
<td>rotational (orthogonal) part of deformation matrix ( x_0^{ij} )</td>
</tr>
<tr>
<td>( S )</td>
<td>coefficient in elastic–plastic constitutive relationship ( (= 3/2 \sigma^2 [1 + (Y/3G)]) )</td>
</tr>
<tr>
<td>( S^{+} ), ( S^{-} )</td>
<td>coefficients generalising element heat-flow expression to external faces</td>
</tr>
<tr>
<td>( s )</td>
<td>thickness of an element</td>
</tr>
<tr>
<td>( s_i )</td>
<td>right-hand side vector of equations obtained by Gaussian elimination</td>
</tr>
<tr>
<td>( \delta^0 )</td>
<td>contravariant component of nominal or Piola–Kirchhoff I stress</td>
</tr>
<tr>
<td>( s_{ij} )</td>
<td>covariant component of nominal or Piola–Kirchhoff I stress</td>
</tr>
<tr>
<td>( T )</td>
<td>absolute temperature. Also tensile stress acting in single-element example</td>
</tr>
<tr>
<td>( T^e )</td>
<td>temperature at end of increment of deformation</td>
</tr>
<tr>
<td>( T^s )</td>
<td>temperature at start of increment of deformation</td>
</tr>
<tr>
<td>( T^b, T^o )</td>
<td>temperature at beginning and end of time step ( \Delta t ). ( T^o ) is also temperature at the centroid of an element</td>
</tr>
<tr>
<td>( T^{i+}, T^{i-} )</td>
<td>temperature at point on ( i )th local curvilinear axis with local coordinate (+1) or (-1)</td>
</tr>
<tr>
<td>([T])</td>
<td>diagonal matrix derived from ([K]) by Gauss–Jordan elimination</td>
</tr>
</tbody>
</table>
\( T^{(0)} \)  estimate of temperature of centroid at end of \( i \)th thermal step

\( T^0 \)  contravariant component of a general tensor in co-ordinate system \( X' \)

\( \Delta T^0 \)  change in temperature during an increment of deformation

\( \Delta T_{\text{est}} \)  estimated change in temperature during an increment of deformation

\( t \)  time

\( t^0 \)  contravariant component of a general tensor in co-ordinate system \( x' \)

\( \rho \)  initial time

\( \delta T \)  change in temperature during one step of thermal calculation

\( \text{d}t \)  infinitesimal time increment

\( \Delta t \)  time interval for increment of deformation

\( U \)  strain energy

\( [U] \)  upper-triangular matrix derived from \([K]\) by Gaussian elimination or Choleski decomposition

\( [U^*] \)  approximation to \([U]\) having a pattern of sparseness similar to that of \([K]\)

\( U_i \)  strain energy of element \( i \)

\( u(x) \)  linear displacement of point from its initial position \( x \) in 1-D example

\( u \)  vector of displacement of a point or sometimes a typical vector

\( u' \)  contravariant component of displacement of a point

\( u_i \)  covariant component of displacement of a point; for Cartesian co-ordinate system, displacement of a point in \( x_i \) direction

\( u_{ij} \)  displacement of a point of element \( i \)

\( u_{i,j} \)  gradient of component \( i \) of displacement in direction \( j \)

\( \dot{u}^{ij} \)  rate of deformation matrix in terms of contravariant components

\( = \frac{d(x^{ij})}{dt} = \dot{x}^{ij} \)

\( \ddot{u}_i \)  rate of deformation matrix in terms of covariant components

\( = \frac{d(x_{i,j})}{dt} = \dot{x}_{i,j} \)

\( \Delta u_{i,j} \)  deformation gradient; change in \( u_{ij} \) during an increment of deformation

\( V \)  volume

\( V^s \)  volume in local element co-ordinate system

\( V^0 \)  contravariant component of \( v \) in co-ordinate system \( X' \)

\( V_i \)  covariant component of \( v \) in co-ordinate system \( X' \)

\( V_i \)  volume of element \( i \)

\( V_{ji} \)  \( j \)th convected component of \( V \)

\( V_{ji} \)  \( j \)th convected component of \( V^i \)

\( v \)  typical vector
Nomenclature

\( v^i \) contravariant component of \( \mathbf{v} \) in co-ordinate system \( x^i \)
\( v_i \) covariant component of \( \mathbf{v} \) in co-ordinate system \( x^i \)
\( \mathbf{v} \) arbitrary infinitesimal vector at \( P \) at time \( t \)
\( \mathbf{v}^i \) arbitrary infinitesimal vector \( \mathbf{v} \) after deformation of \( P \) to \( P' \)
\( v^{ij} \) \( j \)th reference component of \( \mathbf{v} \)
\( v^{-ij} \) \( j \)th reference component of \( \mathbf{v}^{-i} \)
\( W \) work done by external forces
\( W_i \) work done by external forces on element \( i \)
\( w \) multiplication factor in Gaussian elimination procedure
\( \mathbf{X} \) coefficient used in calculation of number of thermal steps
\( \mathbf{X}^i \) co-ordinate \( i \) of a point \( P \) in rotated, convected or transformed axis system (specifically, contravariant component with respect to basis \( G_i \))
\( X_i \) co-ordinate \( i \) of a point \( P \) in rotated, convected or transformed axis system (specifically, contravariant component with respect to basis \( G_i \))
\( \mathbf{X}^{ij} \) co-ordinate \( i \) of a point \( P' \) in rotated, convected or transformed axis system (specifically, contravariant component with respect to basis \( G_i \))
\( x \) linear co-ordinate
\( x^i, x^j \) lower and upper limits of integration in variational expression
\( \mathbf{x} \) position vector of a point
\( \chi^i \) co-ordinate \( i \) of a point \( P \) in initial reference (usually Cartesian) axis system (specifically, contravariant component with respect to basis \( g_i \))
\( x_i \) co-ordinate \( i \) of a point \( P \) in initial reference (usually Cartesian) axis system (specifically, covariant component with respect to basis \( g_i \))
\( x_i^j \) \( j \)th reference co-ordinate of centroid of an element
\( x_i^{+j}, x_i^{-j} \) \( j \)th reference co-ordinate of point on \( j \)th local curvilinear axis with local co-ordinate \( +1 \) or \( -1 \)
\( x^{ij} \) co-ordinate \( i \) of point \( P' \) in initial reference axis system (specifically, contravariant component with respect to basis \( g_i \))
\( x_i^j \) co-ordinate \( i \) of point \( P' \) in initial reference axis system (specifically, covariant component with respect to basis \( g_i \))
\( x_{pi} \) co-ordinate \( i \) of node \( j \) of an element
\( x^{pi} \) contravariant co-ordinate transformation matrix defining deformation of infinitesimal region at point \( P \) at time \( t \) into infinitesimal region at point \( P' \) at time \( t + dt \) \(( = \mathbf{X}^{pi} / \partial x^i / \partial x^{pi})\)
\( x_i^{p ij} \) covariant co-ordinate transformation matrix defining deformation of infinitesimal region at point \( P \) at time \( t \) into infinitesimal region...
Nomenclature

$x^{ij}$
transformation matrix defining a rigid-body rotation; rotational part of $x^{ij}$ ($= r^i$)

$\Delta x'_{ij}$
deformation gradient; change in $x'_{ij}$ during an increment of deformation ($= \Delta u_{ij}$)

$Y$
yield stress in simple tension, a function of strain, strain rate and temperature

$Y^*$
rate of change of $Y$ with respect to plastic strain

$Z$
coefficient used in calculation of number of thermal steps

$\alpha$
angle. Also proportion of work of deformation converted into heat

$\beta$
angle

$\gamma_{ij}$
engineering shear strain in $x_{ij}$ plane ($= 2\epsilon_{ij}^s$, $i \neq j$). Also $\delta_{ij}$, $\delta_{i}^{\gamma_{ij}}$ etc

$\epsilon$
strain

$\epsilon^{AB}$
normal strain in $AB$ direction

$\epsilon^s$
representative strain at start of increment

$\epsilon^e$
representative strain at end of increment calculated by secant-modulus technique ($= \epsilon^e + \Delta \epsilon^e$)

$\epsilon^m$
representative strain during an increment calculated by secant-modulus technique ($= \epsilon^e + \frac{1}{2} \Delta \epsilon^e$)

$\Delta \epsilon^e$
accumulated generalised plastic strain ($= \Delta \epsilon^p$)

$\epsilon_{ij}$
plastic strain at start of plastic part of increment of deformation

$\epsilon^p$
rate of change of plastic strain with respect to time

$\epsilon_i$
vector of 3 normal components of strain and 3 engineering shear components of strain

$\epsilon_i$
principal component of strain

$\epsilon_i$
strain in element $i$

$\epsilon_{ij}$
normal component of strain in $x_i$ direction if $i=j$; shear component of strain in $x_{ij}$ plane if $i \neq j$

$\hat{\epsilon}^e$
contravariant component of strain-rate tensor ($= \frac{1}{2}(\hat{\epsilon}^{ij} + \hat{\epsilon}^{ji})$)

$\hat{\epsilon}^s$
covariant component of strain-rate tensor, equivalent to $\epsilon^e$ for Cartesian (orthogonal) co-ordinate system ($= \frac{1}{2}(\hat{\epsilon}_{ij} + \hat{\epsilon}_{ji})$)

$d\epsilon^p$
deviatoric component of strain ($= \epsilon^p - \delta_{ij}\epsilon^e/3$)

$d\epsilon^p_{ij}$
increment in generalised plastic strain ($= (2d\epsilon^p_{ij}d\epsilon^p_{ij})^{1/2}$)

$d\epsilon^e_{ij}$
elastic part of incremental strain component

$d\epsilon^p_{ij}$
plastic part of incremental strain component
Nomenclature

\[ \Delta \varepsilon^a \] strain increment corresponding to a displacement of \( \Delta d^a \), the first estimate of displacement, in secant-modulus technique

\[ \Delta \varepsilon^b \] strain increment corresponding to a corrected displacement of \( \Delta d^b \), in secant-modulus technique

\[ \Delta \varepsilon^p \] change in plastic strain during an increment of deformation

\[ \Delta \varepsilon^{(0)} \] strain increment calculated from \( \Delta \varepsilon^{(0)} \) in initial-stiffness iteration

\[ \Delta \sigma_0 \] component of LCR increment of strain (specifically, change in strain during plastic part of an increment)

\[ \eta \] coefficient of viscosity

\[ \kappa \] Bulk Modulus of elasticity \( (= E/(3(1-2\nu)) ) \)

\[ d\lambda \] proportionality factor in Prandtl–Reuss elastic–plastic flow equations

\[ \Delta \lambda \] proportionality factor in incremental form of Prandtl–Reuss elastic–plastic flow equations

\[ \nu \] Poisson's Ratio

\[ \sigma \] Cauchy or True stress

\[ \sigma^{AB} \] normal stress acting in \( AB \) direction

\[ \sigma^0 \] representative stress at start of increment

\[ \sigma^e \] representative stress at end of increment corresponding to a strain of \( \varepsilon^e \)

\[ \sigma^h \] hydrostatic component of stress \( (= \sigma_{ii}/3) \)

\[ \sigma^{in} \] representative stress during an increment corresponding to a strain of \( \varepsilon^{in} \)

\[ \bar{\sigma} \] generalised stress \( (= (3\sigma_{ii} \sigma_{ij}/2)^{1/2}) \)

\[ \sigma \] vector of 6 unique components of stress (since \( \sigma_{ij} = \sigma_{ji} \))

\[ \sigma_i \] principal component of stress

\[ \sigma_i \] stress in element \( i \)

\[ \sigma_0 \] normal component of stress in \( x_i \) direction if \( i=j \); shear stress resulting from force in \( x_j \) direction acting on surface with normal in \( x_i \) direction if \( i \neq j \)

\[ \sigma_{ij} \] deviatoric component of stress \( (= \sigma_{ij} - \delta_{ij}\sigma^0) \)

\[ \dot{\sigma}_{ij} \] deviatoric stress at beginning of plastic part of deformation increment

\[ \ddot{\sigma}_{ij} \] deviatoric stress half-way through hypothetical elastic step of deformation increment \( (= \dot{\sigma}_{ij} + \frac{1}{2} \Delta \sigma_{ij}) \)

\[ \bar{\dot{\sigma}}_{ij} \] deviatoric stress half-way through plastic part of step \( (= \dot{\sigma}_{ij} + \frac{1}{2} \Delta \sigma_{ij}) \)

\[ \Delta \sigma^{(0)} \] stress increment calculated from \( \Delta \varepsilon^{(0)} \) in initial-stiffness iteration

\[ \Delta \sigma_{ij} \] change in deviatoric stress during plastic part of deformation increment \( (= \Delta \sigma_{ij} - 2G \Delta \lambda \tilde{\sigma}_{ij}) \)

\[ \Delta \sigma_{ij} \] hypothetical elastic change in stress; change in stress that would
result if plastic part of deformation increment took place elastically

\( = 2G \Delta \varepsilon \)

\( \tau \) maximum permitted error in calculated incremental temperature change

\( \dot{\tau} \) contravariant component of Kirchhoff or Piola–Kirchhoff II stress

\( \dot{\tau} \) time derivative of \( \tau \)

\( \tau^{(b)} \) rotationally-invariant Kirchhoff stress; contravariant component of Kirchhoff stress at a point in a co-ordinate system that rotates with the deforming infinitesimal region at that point

\( \dot{\tau}^{(b)} \) contravariant component of Jaumann rate of Kirchhoff stress

\( \dot{\tau} \) covariant component of Jaumann rate of Kirchhoff stress

\( \phi(\mathbf{x}) \) function of state in a physical system to be determined by variational method

\( \phi, \phi' \) first and second derivatives of \( \phi \) with respect to position

\( \phi(\mathbf{x}) \) approximation to \( \phi \) in the form of linear combination of simple functions

\( \Delta \phi(\mathbf{x}) \) dilatation increment function of position

\( \Delta \phi \) coefficient of linear dilatation function

\( < > \) skew-symmetric part of enclosed matrix: \(<[A]> = (A - A^T)/2\)

\( || \) magnitude of enclosed vector

\( \{ \} \) set of objects

\( \cdot \) scalar product

\( * \) vector product

\( \text{det}(\cdot) \) determinant of enclosed matrix