

Contents

PREFACE	xi
CHAPTER 1: MOTIVATION	
§1	Why bother with measure theory? 1
§2	The cost and benefit of rigor 3
§3	Where to start: probabilities or expectations? 5
§4	The de Finetti notation 7
*§5	Fair prices 11
§6	Problems 13
§7	Notes 14
CHAPTER 2: A MODICUM OF MEASURE THEORY	
§1	Measures and sigma-fields 17
§2	Measurable functions 22
§3	Integrals 26
*§4	Construction of integrals from measures 29
§5	Limit theorems 31
§6	Negligible sets 33
*§7	L^p spaces 36
*§8	Uniform integrability 37
§9	Image measures and distributions 39
§10	Generating classes of sets 41
*§11	Generating classes of functions 43
§12	Problems 45
§13	Notes 51
CHAPTER 3: DENSITIES AND DERIVATIVES	
§1	Densities and absolute continuity 53
*§2	The Lebesgue decomposition 58
§3	Distances and affinities between measures 59
§4	The classical concept of absolute continuity 65
*§5	Vitali covering lemma 68
*§6	Densities as almost sure derivatives 70
§7	Problems 71
§8	Notes 75
CHAPTER 4: PRODUCT SPACES AND INDEPENDENCE	
§1	Independence 77
§2	Independence of sigma-fields 80
§3	Construction of measures on a product space 83
§4	Product measures 88
*§5	Beyond sigma-finiteness 93
§6	SLLN via blocking 95
*§7	SLLN for identically distributed summands 97
*§8	Infinite product spaces 99

§9	Problems	102
§10	Notes	108
CHAPTER 5: CONDITIONING		
§1	Conditional distributions: the elementary case	111
§2	Conditional distributions: the general case	113
§3	Integration and disintegration	116
§4	Conditional densities	118
*§5	Invariance	121
§6	Kolmogorov's abstract conditional expectation	123
*§7	Sufficiency	128
§8	Problems	131
§9	Notes	135
CHAPTER 6: MARTINGALE ET AL.		
§1	What are they?	138
§2	Stopping times	142
§3	Convergence of positive supermartingales	147
§4	Convergence of submartingales	151
*§5	Proof of the Krickeberg decomposition	152
*§6	Uniform integrability	153
*§7	Reversed martingales	155
*§8	Symmetry and exchangeability	159
§9	Problems	162
§10	Notes	166
CHAPTER 7: CONVERGENCE IN DISTRIBUTION		
§1	Definition and consequences	169
§2	Lindeberg's method for the central limit theorem	176
§3	Multivariate limit theorems	181
§4	Stochastic order symbols	182
*§5	Weakly convergent subsequences	184
§6	Problems	186
§7	Notes	190
CHAPTER 8: FOURIER TRANSFORMS		
§1	Definitions and basic properties	193
§2	Inversion formula	195
§3	A mystery?	198
§4	Convergence in distribution	198
*§5	A martingale central limit theorem	200
§6	Multivariate Fourier transforms	202
*§7	Cramér-Wold without Fourier transforms	203
*§8	The Lévy-Cramér theorem	205
§9	Problems	206
§10	Notes	208

Contents

ix

CHAPTER 9: BROWNIAN MOTION

§1	Prerequisites	211
§2	Brownian motion and Wiener measure	213
§3	Existence of Brownian motion	215
*§4	Finer properties of sample paths	217
§5	Strong Markov property	219
*§6	Martingale characterizations of Brownian motion	222
*§7	Functionals of Brownian motion	226
*§8	Option pricing	228
§9	Problems	230
§10	Notes	234

CHAPTER 10: REPRESENTATIONS AND COUPLINGS

§1	What is coupling?	237
§2	Almost sure representations	239
*§3	Strassen's Theorem	242
*§4	The Yurinskii coupling	244
§5	Quantile coupling of Binomial with normal	248
§6	Haar coupling—the Hungarian construction	249
§7	The Komlós-Major-Tusnády coupling	252
§8	Problems	256
§9	Notes	258

CHAPTER 11: EXPONENTIAL TAILS AND THE LAW OF THE ITERATED LOGARITHM

§1	LIL for normal summands	261
§2	LIL for bounded summands	264
*§3	Kolmogorov's exponential lower bound	266
*§4	Identically distributed summands	268
§5	Problems	271
§6	Notes	272

CHAPTER 12: MULTIVARIATE NORMAL DISTRIBUTIONS

§1	Introduction	274
*§2	Fernique's inequality	275
*§3	Proof of Fernique's inequality	276
§4	Gaussian isoperimetric inequality	278
*§5	Proof of the isoperimetric inequality	280
§6	Problems	285
§7	Notes	287

APPENDIX A: MEASURES AND INTEGRALS

§1	Measures and inner measure	289
§2	Tightness	291
§3	Countable additivity	292
§4	Extension to the \cap_c -closure	294
§5	Lebesgue measure	295
§6	Integral representations	296
§7	Problems	300
§8	Notes	300

APPENDIX B: HILBERT SPACES	
§1	Definitions 301
§2	Orthogonal projections 302
§3	Orthonormal bases 303
§4	Series expansions of random processes 305
§5	Problems 306
§6	Notes 306
APPENDIX C: CONVEXITY	
§1	Convex sets and functions 307
§2	One-sided derivatives 308
§3	Integral representations 310
§4	Relative interior of a convex set 312
§5	Separation of convex sets by linear functionals 313
§6	Problems 315
§7	Notes 316
APPENDIX D: BINOMIAL AND NORMAL DISTRIBUTIONS	
§1	Tails of the normal distributions 317
§2	Quantile coupling of Binomial with normal 320
§3	Proof of the approximation theorem 324
§4	Notes 328
APPENDIX E: MARTINGALES IN CONTINUOUS TIME	
§1	Filtrations, sample paths, and stopping times 329
§2	Preservation of martingale properties at stopping times 332
§3	Supermartingales from their rational skeletons 334
§4	The Brownian filtration 336
§5	Problems 338
§6	Notes 338
APPENDIX F: DISINTEGRATION OF MEASURES	
§1	Representation of measures on product spaces 339
§2	Disintegrations with respect to a measurable map 342
§3	Problems 343
§4	Notes 345
INDEX	347