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Introduction and overview

We first introduce the reader to the notions of a system node and an L^p-well-posed linear system with $1 \leq p \leq \infty$, and continue with an overview of the rest of the book.

1.1 Introduction

There are three common ways to describe a finite-dimensional linear time-invariant system in continuous time:

(i) the system can be described in the *time domain* as an *input/output map* $\mathfrak{D}$ from an input signal u into an output signal y;

(ii) the system can be described in the *frequency domain* by means of a *transfer function* $\widehat{\mathfrak{D}}$, i.e., if $\hat{u}$ and $\hat{y}$ are the Laplace transforms of the input u respectively the output y, then $\hat{y} = \widehat{\mathfrak{D}}\hat{u}$ in some right half-plane;

(iii) the system can be described in *state space form* in terms of a set of first order linear differential equations (involving matrices A, B, C, and D of appropriate sizes)

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t) + Du(t), \qquad t \geq 0, \qquad (1.1.1) \\
x(0) &= x_0.
\end{aligned}$$

In (i)–(iii) the input signal u takes its values in the *input space U* and the output signal y takes its values in the *output space Y*, both of which are finite-dimensional real or complex vector spaces (i.e., $\mathbb{R}^k$ or $\mathbb{C}^k$ for some $k = 1, 2, 3, \ldots$), and the state $x(t)$ in (iii) takes its values in the *state space X* (another finite-dimensional vector space).

All of the three descriptions mentioned above are important, but we shall regard the third one, the state space description, as the most fundamental

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one. From a state space description it is fairly easy to get both an input/
output description and a transfer function description. The converse statement
is more difficult (but equally important): to what extent is it true that an input/
output description or a transfer function description can be converted into a
state space description? (Various answers to this question will be given below.)

The same three types of descriptions are used for infinite-dimensional lin-
ear time-invariant systems in continuous time. The main difference is that we
encounter certain technical difficulties which complicate the formulation. As a
result, there is *not just one general infinite-dimensional theory*, but a *collection
of competing theories that partially overlap each other* (and which become
more or less equivalent when specialized to the finite-dimensional case). In this
book we shall concentrate on two quite general settings: the case of a system
which is either *well-posed in an L^p-setting* (for some $p \in [1, \infty]$) or (more
generally), it has a differential description resembling (1.1.1), i.e., it is induced
by a *system node*.

In order to give a definition of a *system node* we begin by combining the
four matrices A, B, C, and D into one single block matrix $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, which
we call the *node* of the system, and rewrite (1.1.1) in the form

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \geq 0, \quad x(0) = x_0. \tag{1.1.2}$$

For a moment, let us ignore the original matrices A, B, C, and D, and simply
regard S as a linear operator mapping $\begin{bmatrix} X \\ U \end{bmatrix}$ into $\begin{bmatrix} X \\ Y \end{bmatrix}$ (recall that we denoted the
input space by U, the state space by X, and the output space by Y). If U, X and
Y are all finite-dimensional, then S is necessarily bounded, but this need not
be true if U, X, or Y is infinite-dimensional. The natural infinite-dimensional
extension of (1.1.1) is to replace (1.1.1) by (1.1.2) and to allow S to be an
unbounded linear operator with some additional properties. These properties
are chosen in such a way that (1.1.2) generates some reasonable family of
trajectories, i.e., for some appropriate class of initial states $x_0 \in X$ and input
functions u the equation (1.1.2) should have a well-defined state trajectory $x(t)$
(defined for all $t \geq 0$) and a well-defined output function y. The set of additional
properties that we shall use in this work is the following.

Definition 1.1.1 We take U, X, and Y to be Banach spaces (sometimes Hilbert
spaces), and call S a *system node* if it satisfies the following four conditions:[1]

(i) S is a closed (possibly unbounded) operator mapping $\mathcal{D}(S) \subset \begin{bmatrix} X \\ U \end{bmatrix}$ into
$\begin{bmatrix} X \\ Y \end{bmatrix}$;

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[1] It follows from Lemma 4.7.7 that this definition is equivalent to the definition of a system node
given in 4.7.2.

(ii) if we split S into $S = \begin{bmatrix} S_X \\ S_Y \end{bmatrix}$ in accordance with the splitting of the range space $\begin{bmatrix} X \\ Y \end{bmatrix}$ (S_X is the 'top row' of S and S_Y is the 'bottom row'), then S_X is closed (with $\mathcal{D}(S_X) = \mathcal{D}(S)$);

(iii) the operator A defined by $Ax = S_X \begin{bmatrix} x \\ 0 \end{bmatrix}$ with domain $\mathcal{D}(A) = \{x \in X \mid \begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathcal{D}(S)\}$ is the generator of a strongly continuous semigroup on X;

(iv) for every $u \in U$ there is some $x \in X$ such that $\begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{D}(S)$.

It turns out that when these additional conditions hold, then (1.1.2) has trajectories of the following type. We use the operators S_X and S_Y defined in (ii) to split (1.1.2) into

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\begin{aligned}
\dot{x}(t) &= S_X \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \geq 0, \quad x(0) = x_0, \\
y(t) &= S_Y \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \geq 0.
\end{aligned}
\tag{1.1.3}
$$

If (i)–(iv) hold, then for each $x_0 \in X$ and $u \in C^2([0, \infty); U)$ such that $\begin{bmatrix} x_0 \\ u(0) \end{bmatrix} \in \mathcal{D}(S)$, there is a unique function $x \in C^1([0, \infty); X)$ (called a *state trajectory*) satisfying $x(0) = x_0$, $\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{D}(S)$, $t \geq 0$, and $\dot{x}(t) = S_X \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$, $t \geq 0$. If we define the output $y \in C([0, \infty); Y)$ by $y(t) = S_Y \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$, $t \geq 0$, then the three functions u, x, and y satisfy (1.1.2) (this result is a slightly simplified version of Lemma 4.7.8).

Another consequences of conditions (i)–(iv) above is that it is almost (but not quite) possible to split a system node S into $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ as in the finite-dimensional case. If X is finite-dimensional, then the operator A in (iii) will be bounded, and this forces the full system node S to be bounded, with $\mathcal{D}(S) = \begin{bmatrix} X \\ U \end{bmatrix}$. Trivially, in this case S can be decomposed into four bounded operators $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. If X is infinite-dimensional, then a partial decomposition still exists. The operator A in this partial decomposition corresponds to an extension $A_{|X}$ of the *semigroup generator A* in (iii).[2] This extension is defined on all of X, and it maps X into a larger 'extrapolation space' X_{-1} which contains X as a dense subspace. There is also a *control operator B* which maps U into X_{-1}, and the operator S_X defined in (ii) (the 'top row' of X) is the restriction to $\mathcal{D}(S)$ of the operator $\begin{bmatrix} A_{|X} & B \end{bmatrix}$ which maps $\begin{bmatrix} X \\ U \end{bmatrix}$ into X_{-1}. (Furthermore, $\mathcal{D}(S) = \{\begin{bmatrix} x \\ u \end{bmatrix} \in \begin{bmatrix} X \\ U \end{bmatrix} \mid \begin{bmatrix} A_{|X} & B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \in X\}$.) Thus, S_X always has a decomposition (after an appropriate extension of its domain and also an extension of the range space). The 'bottom row' S_Y is more problematic, due to the fact that it is not always possible to embed Y as a dense subspace in some larger space Y_{-1} (for example, Y may be finite-dimensional). It is still true,

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[2] We shall also refer to A as the *main operator* of the system node.

however, that it is possible to define an *observation operator C* with domain
$\mathcal{D}(C) = \mathcal{D}(A)$ by $Cx = S_Y \begin{bmatrix} x \\ 0 \end{bmatrix}$, $x \in \mathcal{D}(A)$. The *feedthrough operator D* in
the finite-dimensional decomposition $A = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ need not always exist, and it
need not be unique. However, this lack of a unique well-defined feedthrough
operator is largely compensated by the fact that every system node has a *transfer
function*, defined on the resolvent set of the operator A in (iii). See Section 4.7 for
details.[3]

The other main setting that we shall use (and after which this book has
been named) is the L^p-*well-posed setting* with $1 \leq p \leq \infty$. This setting can be
introduced in two different ways. One way is to first introduce a system node
of the type described above, and then add the requirement that for all $t > 0$, *the
final state $x(t)$ and the restriction of y to the interval* $[0, t)$ *depend continuously
on x_0 and the restriction of u to* $[0, t)$. This added requirement will give us an
L^p-well-posed linear system if we use the X-norm for x_0 and $x(t)$, the norm in
$L^p([0, t); U)$ for u, and the norm in $L^p([0, t); Y)$ for y.[4] (See Theorem 4.7.13
for details.)

However, it is also possible to proceed in a different way (as we do in
Chapter 2) and to introduce the notion of an L^p-well-posed linear system *without
any reference to a system node*. In this approach we look directly at the mapping
from the initial state x_0 and the input function (restricted to the interval $[0, t)$)
to the final state $x(t)$ and the output function y (also restricted to the interval
$[0, t)$). Assuming the same type of continuous dependence as we did above, the
relationship between these four objects can be written in the form (we denote
the restrictions of u and y to some interval $[s, t)$ by $\pi_{[s,t)}u$, respectively $\pi_{[s,t)}y$)

$$\begin{bmatrix} x(t) \\ \pi_{[0,t)}y \end{bmatrix} = \left[\begin{array}{c|c} \mathfrak{A}_0^t & \mathfrak{B}_0^t \\ \hline \mathfrak{C}_0^t & \mathfrak{D}_0^t \end{array} \right] \begin{bmatrix} x_0 \\ \pi_{[0,t)}u \end{bmatrix}, \quad t \geq 0,$$

for some families of bounded linear operator $\mathfrak{A}_0^t \colon X \to X$, $\mathfrak{B}_0^t \colon L^p([0, t); U) \to$
X, $\mathfrak{C}_0^t \colon X \to L^p([0, t); Y)$, and $\mathfrak{D}_0^t \colon L^p([0, t); U) \to L^p([0, t); Y)$. If these
families correspond to the trajectories of some system node (as described ear-
lier), then they necessarily *satisfy some algebraic conditions*, with can be stated
without any reference to the system node itself. Maybe the simplest way to list
these algebraic conditions is to look at a slightly extended version of (1.1.2)

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[3] Another common way of constructing a system node is the following. Take any semigroup
generator A in X, and extend it to an operator $A_{|X} \in \mathcal{B}(X; X_{-1})$. Let $B \in \mathcal{B}(U; X_{-1})$ and
$C \in \mathcal{B}(X_1; U)$ be arbitrary, where X_1 is $\mathcal{D}(A)$ with the graph norm. Finally, fix the value of the
transfer function to be a given operator in $\mathcal{B}(U; Y)$ at some arbitrary point in $\rho(A)$, and use
Lemma 4.7.6 to construct the corresponding system node.

[4] Here we could just as well have replaced the interval $[0, t)$ by $(0, t)$ or $[0, t]$. However, we shall
later consider functions which are defined pointwise everywhere (as opposed to almost
everywhere), and then it is most convenient to use half-open intervals of the type $[s, t)$, $s < t$.

where the initial time zero has been replaced by a general initial time s, namely

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \geq s, \quad x(s) = x_s, \tag{1.1.4}$$

and to also look at the corresponding maps from x_s and $\pi_{[s,t)}u$ to $x(t)$ and $\pi_{[s,t)}y$ which we denote by

$$\begin{bmatrix} x(t) \\ \pi_{[s,t)}y \end{bmatrix} = \left[\begin{array}{c|c} \mathfrak{A}_s^t & \mathfrak{B}_s^t \\ \hline \mathfrak{C}_s^t & \mathfrak{D}_s^t \end{array} \right] \begin{bmatrix} x_s \\ \pi_{[s,t)}u \end{bmatrix}, \quad s \leq t.$$

These two-parameter families of bounded linear operators $\mathfrak{A}_s^t$, $\mathfrak{B}_s^t$, $\mathfrak{C}_s^t$, and $\mathfrak{D}_s^t$ have the properties listed below. In this list of properties we denote the left-shift operator by

$$(\tau^t u)(s) = u(t + s), \quad -\infty < s, t < \infty,$$

and the identity operator by 1.

Algebraic conditions 1.1.2 *The operator families $\mathfrak{A}_s^t$, $\mathfrak{B}_s^t$, $\mathfrak{C}_s^t$, and $\mathfrak{D}_s^t$ satisfy the following conditions:*[5]

(i) *For all $t \in \mathbb{R}$,*

$$\left[\begin{array}{c|c} \mathfrak{A}_t^t & \mathfrak{B}_t^t \\ \hline \mathfrak{C}_t^t & \mathfrak{D}_t^t \end{array} \right] = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array} \right].$$

(ii) *For all $s \leq t$,*

$$\left[\begin{array}{c|c} \mathfrak{A}_s^t & \mathfrak{B}_s^t \\ \hline \mathfrak{C}_s^t & \mathfrak{D}_s^t \end{array} \right] = \left[\begin{array}{c|c} \mathfrak{A}_s^t & \mathfrak{B}_s^t \pi_{[s,t)} \\ \hline \pi_{[s,t)} \mathfrak{C}_s^t & \pi_{[s,t)} \mathfrak{D}_s^t \pi_{[s,t)} \end{array} \right].$$

(iii) *For all $s \leq t$ and $h \in \mathbb{R}$,*

$$\left[\begin{array}{c|c} \mathfrak{A}_{s+h}^{t+h} & \mathfrak{B}_{s+h}^{t+h} \\ \hline \mathfrak{C}_{s+h}^{t+h} & \mathfrak{D}_{s+h}^{t+h} \end{array} \right] = \left[\begin{array}{c|c} \mathfrak{A}_s^t & \mathfrak{B}_s^t \tau^h \\ \hline \tau^{-h} \mathfrak{C}_s^t & \tau^{-h} \mathfrak{D}_s^t \tau^h \end{array} \right].$$

(iv) *For all $s \leq r \leq t$,*

$$\left[\begin{array}{c|c} \mathfrak{A}_s^t & \mathfrak{B}_s^t \\ \hline \mathfrak{C}_s^t & \mathfrak{D}_s^t \end{array} \right] = \left[\begin{array}{c|c} \mathfrak{A}_r^t \mathfrak{A}_s^r & \mathfrak{B}_r^t + \mathfrak{A}_r^t \mathfrak{B}_s^r \\ \hline \mathfrak{C}_r^t \mathfrak{A}_s^r + \mathfrak{C}_s^r & \mathfrak{D}_r^t + \mathfrak{C}_r^t \mathfrak{B}_s^r + \mathfrak{D}_s^r \end{array} \right].$$

All of these conditions have natural interpretations (see Sections 2.1 and 2.2 for details): (i) is an initial condition, (ii) says that the system is causal, (iii)

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[5] By Theorem 2.2.14, these algebraic conditions are equivalent to those listed in Definition 2.2.1.

says that the system is time-invariant, and (iv) gives a formula for how to patch two solutions together, the first of which is defined on $[s, r]$ and the second on $[r, t]$, and with the initial state of the second solution equal to the final state of the first solution at the 'switching time' r. For example, if we take a closer look at the family $\mathfrak{A}_s^t$, then (iii) says that $\mathfrak{A}_s^t = \mathfrak{A}_0^{t-s}$ for all $s \leq t$, (i) says that $\mathfrak{A}_0^0 = 1$, and (iv) says that $\mathfrak{A}_0^t = \mathfrak{A}_0^r \mathfrak{A}_0^{t-r}$ for all $0 \leq r \leq t$. This means that the family $\mathfrak{A}_0^t$ is simply a semigroup (it is the semigroup generated by the operator A of the corresponding system node).

Not only are the conditions (i)–(iv) above *necessary* for the family $\begin{bmatrix} \mathfrak{A}_s^t & \mathfrak{B}_s^t \\ \mathfrak{C}_s^t & \mathfrak{D}_s^t \end{bmatrix}$ to be generated by a system node S through the equation (1.1.4), but they are *sufficient* as well (when combined with the appropriate continuity assumptions). This will be shown in Chapters 3 and 4 (out of which the former deals exclusively with semigroups). However, it is possible to develop a fairly rich theory by simply appealing to the algebraic conditions (i)–(iv) above (and appropriate continuity conditions), without any reference to the corresponding system node. Among other things, every L^p-well-posed linear system has a finite growth bound, identical to the growth bound of its semigroup $\mathfrak{A}_0^t$. See Chapter 2 for details.

Most of the remainder of the book deals with extensions of various notions known from the theory of finite-dimensional systems to the setting of L^p-well-posed linear systems, and even to systems generated by arbitrary system nodes. Some of the extensions are straightforward, others are more complicated, and some finite-dimensional results are simply not true in an infinite-dimensional setting. Conversely, many of the infinite-dimensional results that we present do not have any finite-dimensional counterparts, in the sense that these statements become trivial if the state space is finite-dimensional. In many places the case $p = \infty$ is treated in a slightly different way from the case $p < \infty$, and the class of L^∞-well-posed linear systems is often replaced by another class of systems, the *Reg*-well-posed class, which allows functions to be evaluated everywhere (recall that functions in L^∞ are defined only almost everywhere), and which restricts the set of permitted discontinuities to jump discontinuities.

The last three chapters have a slightly different flavor from the others. We replace the general Banach space setting which has been used up to now by a standard Hilbert space setting, and explore some connections between well-posed linear systems, Fourier analysis, and operator theory. In particular, in Section 10.3 we establish the standard connection between the class of bounded time-invariant causal operators on L^2 and the set of bounded analytic functions on the right half-plane, and in Sections 10.5–10.7 the admissibility and boundedness of scalar control and observation operators for contraction semigroups are characterized by means of the Carleson measure theorem. Chapter 11 has

a distinct operator theory flavor. It contains among others a systems theory interpretation of the basic dilation and model theory for continuous-time contraction semigroups on Hilbert spaces.

Chapter 12 contains a short introduction to discrete-time systems (and it also contains a section on continuous-time systems). Some auxiliary results have been collected in the appendix.

After this rough description of what this book is all about, let us also tell the reader what this book is *not* about, and give some indications of where to look for these missing results.

There are a number of examples of L^p-well-posed linear system given in this book, but these are primarily of a mathematical nature, and they are not the true physical examples given in terms of partial differential equations which are found in books on mathematical physics. There are two reasons for this lack of physical examples. One of them is the lack of space and time. The present book is quite large, and any addition of such examples would require a significant amount of additional space. It would also require another year or two or three to complete the manuscript. The other reason is that the two recent volumes Lasiecka and Triggiani (2000a, b) contain an excellent collection of examples of partial differential equations modeling various physical systems. By Theorem 5.7.3(iii), most of the examples in the first volume dealing with parabolic problems are *Reg*-well-posed. Many of the examples in the second volume dealing with hyperbolic problems are L^2-well-posed. Almost all the examples in Lasiecka and Triggiani (2000a, b) are generated by system nodes. (The emphasis of these two volumes is quite different from the emphasis of this book. They deal with *optimal control*, whereas we take a more general approach, focusing more on input/output properties, transfer functions, coprime fractions, realizations, passive and conservative systems, discrete time systems, model theory, etc.)

Our original main motivation for introducing the class of systems generated by arbitrary system nodes was that this class is a very natural setting for a study of *impedance passive* systems. Such systems need not be well-posed, but under rather weak assumptions they are generated by system nodes. The decision *not* to include a formal discussion of impedance passive systems in this book was not easy. Once more this decision was dictated partly by the lack of space and time, and partly by the fact that there is another recently discovered setting which may be even more suitable for this class of systems, namely the continuous time analogue of the state/signal systems introduced in Arov and Staffans (2004, see also Ball and Staffans 2003). Impedance passive systems are discussed in the spirit of this book in Staffans (2002a, b, c).

Another obvious omission (already mentioned above) is the lack of results concerning quadratic optimal control. This omission may seem even more

strange in light of the fact that the original motivation for writing this book was to present a general theory that could be used in the study of optimal control problems (of definite and indefinite type). However, also this omission has two valid reasons. The first one is the same as we mentioned above, i.e., lack of space and time. The other reason is even more fundamental: the theory of optimal control is at this very moment subject to very active research, and it has not yet reached the needed maturity to be written down in the form of a monograph. We are here thinking about a general theory in the spirit of this book. There do, of course, exist quite mature theories for various subclasses of systems. One such class is the one which assumes that the system is of the 'classical' form (1.1.1), where A is the generator of a strongly continuous semigroup and the operators B, C, and D are bounded. This class is thoroughly investigated in Curtain and Zwart (1995). Systems of this type are easy to deal with (hence, they have a significant pedagogical value), but they are too limited to cover many of the interesting boundary control systems encountered in mathematical physics. (For example, the models developed in Sections 11.6 and 11.7 have bounded B, C, and D only in very special cases.) Other more general (hence less complete) theories are found in, e.g., Lions (1971), Curtain and Pritchard (1978), Bensoussan *et al.* (1992), Fattorini (1999), and Lasiecka and Triggiani (2000a, b). Quadratic optimal control results in the setting of L^2-well-posed linear systems are found in Mikkola (2002), Staffans (1997, 1998a, b, c, d), Weiss (2003), and Weiss and Weiss (1997).

There is a significant overlap between some parts of this book and certain books which deal with 'abstract system theory', such as Fuhrmann (1981) and Feintuch and Saeks (1982), or with operator theory, such as Lax and Phillips (1967), Sz.-Nagy and Foiaş (1970), Brodskiǐ (1971), Livšic (1973), and Nikol'skiǐ (1986). In particular, Chapter 11 can be regarded as a natural continuous-time analogue of one of the central parts of Sz.-Nagy and Foiaş (1970, rewritten in the language of L^2-well-posed linear systems).

1.2 Overview of chapters 2–13

Chapter 2 In this chapter we develop the basic theory of L^p-well-posed linear systems starting from a set of algebraic conditions which is equivalent to 1.1.2. We first simplify the algebraic conditions 1.1.2 by using a part of those conditions to replace the original two-parameter families $\mathfrak{A}_s^t$, $\mathfrak{B}_s^t$, $\mathfrak{C}_s^t$, and $\mathfrak{D}_s^t$ introduced in Section 1.1 by a semigroup $\mathfrak{A}^t$, $t \geq 0$, and three other operators, the *input map* $\mathfrak{B} = \mathfrak{B}_{-\infty}^0$, the *output map* $\mathfrak{C} = \mathfrak{C}_0^\infty$, and the *input/output map* $\mathfrak{D} = \mathfrak{D}_{-\infty}^\infty$. The resulting algebraic conditions that $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, and $\mathfrak{D}$ have to satisfy are listed in 2.1.3 and again in Definition 2.2.1. The connection between the

quadruple $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, and $\mathfrak{D}$ and the original four operator families $\mathfrak{A}_s^t$, $\mathfrak{B}_s^t$, $\mathfrak{C}_s^t$, and $\mathfrak{D}_s^t$ is explained informally in Section 2.1 and more formally in Definition 2.2.6 and Theorem 2.2.14. Thus, we may either interpret an L^p-well-posed linear system as a quadruple $\Sigma = \left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$, or as a two-parameter family of operators $\Sigma_s^t = \left[\begin{smallmatrix} \mathfrak{A}_s^t & \mathfrak{B}_s^t \\ \mathfrak{C}_s^t & \mathfrak{D}_s^t \end{smallmatrix}\right]$, where s represents the initial time and t the final time.

In the case where $p = \infty$ we often require the system to be *Reg*-well-posed instead of L^∞-well-posed. Here *Reg* stands for the class of *regulated functions* (which is described in more detail in Section A.1). By a regulated function we mean a function which is locally bounded, right-continuous, and which has a left-hand limit at each finite point. The natural norm in this space is the L^∞-norm (i.e., the sup-norm). In this connection we introduce the following terminology (see Definition 2.2.4). By an $L^p|Reg$-*well-posed linear system* we mean a system which is either *Reg*-well-posed or L^p-well-posed for some p, $1 \le p \le \infty$, and by a *well-posed linear system* we mean a system which is either *Reg*-well-posed or L^p-well-posed for some p, $1 \le p < \infty$. Thus, the L^p-case with $p = \infty$ is included in the former class but not in the latter. The reason for this distinction is that not all results that we present are true for L^∞-well-posed systems. Whenever we write $L^p|Reg$ we mean either L^p or *Reg*, whichever is appropriate at the moment.

In our original definition of the operators $\mathfrak{B}$ and $\mathfrak{D}$ we restrict their domains to consist of those input functions which are locally in $L^p|Reg$ with values in U, and whose supports are bounded to the left. The original range spaces of $\mathfrak{C}$ and $\mathfrak{D}$ consist of output functions which are locally in $L^p|Reg$ with values in Y. However, as we show in Theorem 2.5.4, every $L^p|Reg$-well-posed linear system has a *finite exponential growth bound* (equal to the growth bound of its semigroup). This fact enables us to extend the operators $\mathfrak{B}$ and $\mathfrak{D}$ to a larger domain, and to confine the ranges of $\mathfrak{C}$ and $\mathfrak{D}$ to a smaller space. More precisely, we are able to relax the original requirement that the support of the input function should be bounded to the left, replacing it by the requirement that the input function should belong to some exponentially weighted $L^p|Reg$-space. We are also able to show that the ranges of $\mathfrak{C}$ and $\mathfrak{D}$ lie in an exponentially weighted $L^p|Reg$-space (the exponential weight is the same in both cases, and it is related to the growth bound of the system). In later discussions we most of the time use these extended/confined versions of $\mathfrak{B}$, $\mathfrak{C}$, and $\mathfrak{D}$.

As part of the proof of the fact that every $L^p|Reg$-well-posed linear system has a finite growth bound we show in Section 2.4 that every such system can be interpreted as a discrete-time system $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$ with infinite-dimensional input and output spaces, and with bounded operators $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, and $\mathbb{D}$. More precisely, $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathfrak{A}_0^T & \mathfrak{B}_0^T \\ \mathfrak{C}_0^T & \mathfrak{D}_0^T \end{smallmatrix}\right]$, the discrete-time input space is $L^p|Reg([0, T); U)$, and the output space is $L^p|Reg([0, T); Y)$, for some $T > 0$. We achieve

this by regarding $L^p|Reg([0, \infty); U)$ as an infinite product of the spaces $L^p|Reg([kT, (k + 1)T); U)$, $k = 0, 1, 2, \ldots$, and treating $L^p|Reg([0, \infty); Y)$ in a similar manner.

In Section 2.6 we show that a linear time-invariant causal operator which maps $L^p|Reg_{\mathrm{loc}}([0, \infty); U)$ into $L^p|Reg_{\mathrm{loc}}([0, \infty); U)$ can be interpreted as the input/output map of some $L^p|Reg$-well-posed linear system if and only if it is exponentially bounded. In Section 2.7 we show how to re-interpret an L^p-well-posed linear system with $p < \infty$ as a strongly continuous semigroup in a suitable (infinite-dimensional) state space. This construction explains the connection between a well-posed linear system and the semigroups occurring in scattering theory studied in, e.g., Lax and Phillips (1967).

Chapter 3 Here we develop the basic theory of C_0 (i.e., strongly continuous) *semigroups and groups*. The treatment resembles the one found in most textbooks on semigroup theory (such as Pazy (1983)), but we put more emphasis on certain aspects of the theory than what is usually done. The generator of a C_0 semigroup and its resolvent are introduced in Section 3.2, and the celebrated Hille–Yosida generating theorem is stated and proved in Section 3.4, together with theorems characterizing generators of contraction semigroups. The primary examples are shift semigroups in (exponentially weighted) L^p-spaces. Dual semigroups are studied in Section 3.5, both in the reflexive case and the nonreflexive case (in the latter case the dual semigroup is defined on a closed subspace of the dual of the original state space). Here we also explain the duality concept which we use throughout the whole book: in spite of the fact that most of the time we work in a Banach space instead of a Hilbert space setting, we still use the conjugate-linear dual rather than the standard linear dual (to make the passage from the Banach space to the Hilbert space setting as smooth as possible).

The first slightly nonstandard result in Chapter 3 is the introduction in Section 3.6 of "Sobolev spaces" with positive and negative index induced by a semigroup generator A, or more generally, by an unbounded densely defined operator A with a nonempty resolvent set.[6] If we denote the original state space by $X = X_0$, then this is a family of spaces

$$\cdots \subset X_2 \subset X_1 \subset X \subset X_{-1} \subset X_{-2} \subset \cdots,$$

where each embedding is continuous and dense, and $(\alpha - A)$ maps X_{j+1} one-to-one onto X_j for all α in the resolvent set of A and all $j \geq 0$. A similar statement is true for $j < 0$: the only difference is that we first have to *extend A*

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[6] In the Russian tradition these spaces are known as spaces with a 'positive norm' respectively 'negative norm'. Spaces with positive index are sometimes referred to as 'interpolation spaces,' and those with negative index as 'extrapolation spaces'.