The Theory of Complex Angular Momenta

Gribov Lectures on Theoretical Physics

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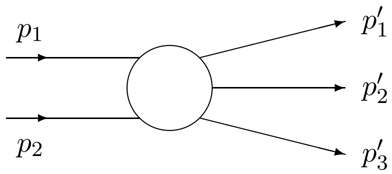
In these lectures the theory of complex angular momenta is presented. It is assumed that readers are familiar with the methods of modern quantum field theory (QFT). Nevertheless we shall briefly recall its basic principles.

1.1 Basic principles

The main experimental fact underlying the theory is the existence of strong interactions between particles of non-zero masses. The theory is constructed for quantities which have a direct physical meaning.

1.1.1 Invariant scattering amplitude and cross section

Such quantities are the scattering amplitudes,



which are supposed to be functions of the kinematical invariants only: $A(p_1, \ldots, p_n) = A(p_i^2, p_i p_k)$. For simplicity, let us begin by considering the scattering of neutral, spinless particles as shown in Fig. 1.1. We use a normalization of the scattering amplitudes such that the kinematical factors arising from the wave functions of the external particles are factorized out. The cross section of any process can be defined in terms of

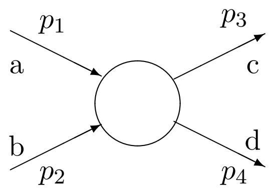


Fig. 1.1. Two-particle scattering

the invariant amplitude A as follows:

$$\begin{aligned} d\sigma_n &= (2\pi)^4 \delta\left(p_1 + p_2 - \sum_i p'_i\right) |A|^2 \prod_{i=1}^{n} \frac{d^3 p'_i}{2p'_{i0}(2\pi)^3} \frac{1}{I}, \\ I &= 4p_{10}p_{20}J = 4\sqrt{(p_1 p_2)^2 - m_1^2 m_2^2}\,. \end{aligned} \tag{1.1}$$

Here the factor $(2\pi)^4\delta()$ originates from energy–momentum conservation, $d^3p'_i/2p'_{i0}(2\pi)^3$ from the phase space volume; I is the Møller factor which combines the flux density J of the initial particles and $(2p_{10}\,2p_{20})^{-1}$ coming from their wave functions.

1.1.2 Analyticity and causality

It is assumed that the scattering amplitude A is an analytic function of its arguments (for instance it cannot contain terms like $\Theta(p_{i0})$). This assumption is a manifestation of the causality principle. Without analyticity, the scattered waves could appear at their source before being emitted. Additionally, it is natural to conjecture at this point that the growth of the scattering amplitude, as one of the invariants tends to infinity for fixed values of the remaining invariants, is polynomially bounded,

$$|A(p_1, \ldots, p_n)| < (p_i p_j)^N.$$

This assumption is closely related to causality and the locality of the interaction. One needs it in order to write the dispersion representation for the amplitudes (to be able to close the integration contour over an infinitely large circle).

1.1.3 Singularities

It is also assumed that all singularities of the amplitude on the physical sheet have the meaning of reaction thresholds, i.e. they are determined by physical masses of the intermediate state particles. In terms of Feynman diagrams they are the Landau singularities.

1.1.4 Crossing symmetry

We will clarify the meaning of crossing, taking as an example a four-particle amplitude. Since this amplitude depends on the kinematical invariants (and not on the sign of p_{i0}), the same analytic function describes the reaction

$$a(p_1) + b(p_2) \to c(p_3) + d(p_4) \qquad \text{for} \quad p_{10}, p_{20}, p_{30}, p_{40} > 0$$

as well as

$$a(p_1) + \bar{c}(-p_3) \to \bar{b}(-p_2) + d(p_4) \qquad \text{for} \quad p_{10}, p_{40} > 0,\ \ p_{20}, p_{30} < 0$$

and

$$a(p_1) + \bar{d}(-p_4) \to \bar{b}(-p_2) + c(p_3) \qquad \text{for} \quad p_{10}, p_{30} > 0,\ \ p_{20}, p_{40} < 0\,.$$

For an unstable particle, there is the additional reaction $a \to \bar{b} + c + d$ $(p_{10}, p_{30}, p_{40} > 0, p_{20} < 0)$.

In fact, the crossing symmetry implies the CPT-theorem – invariance of the amplitude A with respect to the combination of charge conjugation C, space reflection P and time reversal T.

Crossing symmetry follows from the first three assumptions. It can be shown that the same assumptions allow us to prove the spin-statistics relation theorem (the Pauli theorem).

1.1.5 The unitarity condition for the scattering matrix

Unitarity has a simple physical meaning: the sum of probabilities of all processes which are possible at a given energy is equal to unity, $SS^+ = 1$. If $S = 1 + \mathrm{i}\,A$, then

$$\mathrm{i}\,(A - A^+) = -AA^+.$$

Representing the amplitude A as the sum of its real and imaginary parts, $A = \mathrm{Re}\,A + \mathrm{i}\,\mathrm{Im}\,A$, the unitarity condition takes the form

$$2\,\mathrm{Im}\,A = AA^+. \tag{1.2}$$

1.2 Mandelstam variables for two-particle scattering

Let us show how all the above principles work in the case of the four-particle amplitude.

Although the amplitude of the $2 \to 2$ process depends evidently on two independent variables, that is the energy of the incoming particles and the scattering angle, it is more convenient to consider A as a function of three Mandelstam variables

$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2 .$$

They are related to each other by

$$s + t + u = \sum_{i=1}^{4} m_i^2$$

where the sum runs over the masses of all particles participating in the collision.

For the sake of simplicity, in what follows we restrict ourselves to the case of equal particle masses, $m_i = \mu$.

The Mandelstam variables have a simple physical meaning. For instance, in the centre-of-mass system (cms) of the reaction $a + b \to c + d$ (the so-called s-channel), s is the square of the total energy of the colliding particles and $t = -(\boldsymbol{p}_1 - \boldsymbol{p}_3)^2$ is the square of the momentum transfer from a to c. In the cms of the reaction $a + \bar{c} \to \bar{b} + d$ (t-channel), t plays the role of the total energy squared, and s is the square of momentum transfer. The variables u and t, respectively, play similar rôles in the u-channel reaction $a + \bar{d} \to \bar{b} + c$.

1.2.1 The Mandelstam plane

It is convenient, following Landau, to represent the kinematics of the three reactions graphically on the Mandelstam plane. We use here the well known geometrical fact that the sum of the distances from a point on the plane to the sides of an equilateral triangle does not depend on the position of the point. Therefore, taking into account the condition $s + t + u = 4\mu^2$, let us measure s, t and u as the distances to the sides of the triangle.

It is easy then to represent the physical region of any reaction on such a plane. For instance, the physical region of the reaction $a + \bar{c} \to \bar{b} + d$ corresponds to $t \geq 4\mu^2$, $s \leq 0$, $u \leq 0$ and it is shown on Fig. 1.2 as the upper shaded area. The physical regions of the other reactions can be identified in a similar manner.

In the case of the scattering of *identical* neutral particles the amplitude in each physical region is the same and it satisfies the unitarity condition separately in each region.

Examining the Mandelstam plane Fig. 1.2 we notice an interesting feature: as we move from positive to negative values of s (from the physical

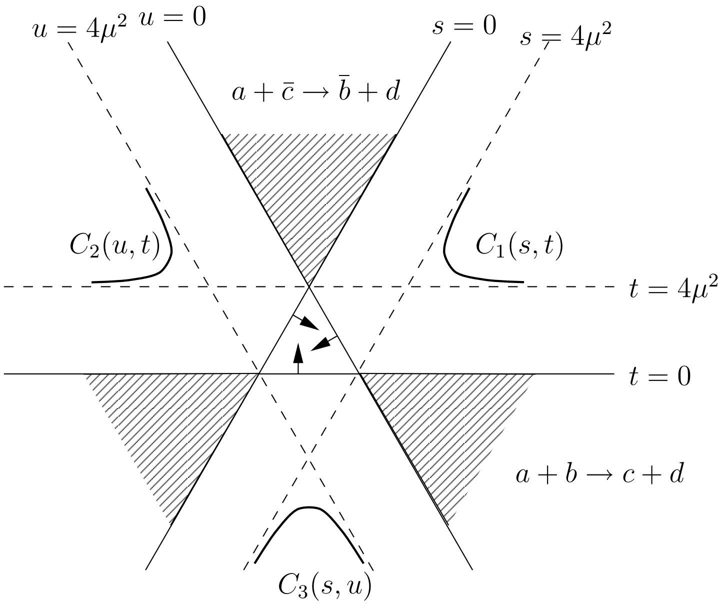


Fig. 1.2. Crossing reactions on the Mandelstam plane

region of the s-channel to the u-channel), the energy dependence of the scattering amplitude turns into the angular dependence.

1.2.2 Threshold singularities on the Mandelstam plane

Let us discuss now singularities of the amplitude. As an illustration, we consider elastic scattering of neutral pions: $\pi^0 + \pi^0 \to \pi^0 + \pi^0$. We will assume that (in accordance with experiment) pions are the lightest stable hadrons and that there is no bound state of two neutral pions. Then, the amplitude has no singularities at $s < 4\mu^2$. The first threshold lies at $s = (2\mu)^2$. It corresponds to the two-particle intermediate state. The next, three-particle threshold could have appeared at $s = (3\mu)^2$. In reality, the second threshold in the pion scattering amplitude is situated at $s = (4\mu)^2$ – the four-particle state, since the transition of two pions into three is forbidden by G-parity conservation.

Similar singularities in *energy* are known to appear in quantum mechanics, for instance the threshold singularity at $s \to 4\mu^2$.

There is however a principal difference between relativistic and non-relativistic theories in the interpretation of the singularities in *momentum transfer*.

In quantum mechanics such singularities are determined by the potential. For instance, the Yukawa potential

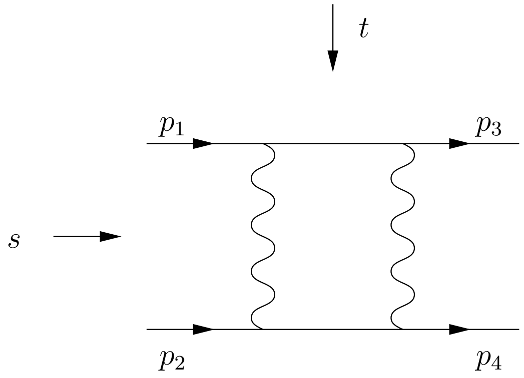
$$V(r) \propto \frac{\exp(-\alpha r)}{r}$$

corresponds to a pole of the scattering amplitude in the plane of the squared momentum transfer k:

$$A(k^2) \propto \frac{1}{k^2 + \alpha^2}.$$

In the relativistic theory the rôle of the potential is played by energy singularities in the t-channel, thresholds at $t = 4\mu^2$, $16\mu^2$ and so on.

Let us illustrate this statement by considering the box diagram



whose contribution we may interpret as defining the potential in the *next-to-Born* approximation. It is easy to see that the radius of this potential is $r = 1/2\mu$.

Thus, the assumption that all the singularities of the scattering amplitude are determined by the masses of real particles implies that there are no potentials with an infinite radius (since all hadrons have non-zero masses).

1.3 Partial wave expansion and unitarity

In order to obtain more concrete results, we must exploit analyticity and unitarity of the S-matrix.

Due to conservation of angular momentum, the unitarity condition for scattering amplitudes with given angular momentum ℓ becomes diagonal. It is convenient, therefore, to expand the s-channel amplitude into partial waves:

$$A(s,t) = \sum_{\ell=0}^{\infty} f_\ell(s)(2\ell+1)P_\ell(z), \tag{1.3a}$$

where $P_\ell(z)$ is the Legendre polynomial and z is the cosine of the scattering angle:

$$z = \cos\Theta_s = 1 + \frac{2t}{s-4\mu^2} = \frac{u-t}{u+t}\,. \tag{1.3b}$$

From (1.3b) it becomes obvious that in the physical region of s-channel $(t, u \le 0)$ we have $-1 \le z \le 1$, as expected.

Substituting the expansion (1.3a) into (1.2) and using well known orthogonality properties of Legendre polynomials, it is straightforward to derive the unitarity condition for partial amplitudes $f_\ell(s)$. It acquires a particularly simple form*

$$\mathrm{Im}\, f_\ell(s) \;=\; \frac{k_s}{16\pi\omega_s}\, f_\ell(s) f_\ell^*(s) \,+\, \Delta, \tag{1.4a}$$

where p and ω stand for cms particle momentum and energy, respectively,

$$k_s = \frac{\sqrt{s-4\mu^2}}{2}, \qquad \omega_s = \frac{\sqrt{s}}{2}. \tag{1.4b}$$

In (1.4a) Δ represents the contribution of the inelastic channels, $\Delta > 0$. The elastic case, $\Delta = 0$, can be solved explicitly:

$$f_\ell(s) = \mathrm{i}\,\frac{8\pi}{v}\left[1 - \mathrm{e}^{2\mathrm{i}\,\delta_\ell(s)}\right], \qquad v = \frac{k_s}{\omega_s}, \tag{1.5a}$$

with δ_l the scattering phase.

The solution of the elastic unitarity condition has the same form as in non-relativistic quantum mechanics except for the velocity factor $v = k/\omega$ which arises due to relativistic normalization of the amplitude A.

In the general case the solution of (1.4a) can be parametrized with the help of the 'elasticity parameter' $\eta_\ell(s) \le 1$:

$$f_\ell(s) = \mathrm{i}\,\frac{8\pi}{v}\left[1 - \eta_\ell \cdot \mathrm{e}^{2\mathrm{i}\,\delta_\ell}\right], \qquad \eta_\ell^2 = 1 - \frac{v}{4\pi}\Delta. \tag{1.5b}$$

From (1.5) it follows that partial wave amplitudes are bounded from above:

$$\mathrm{Im}\, f_\ell \le |f_\ell| \;\le\; 16\pi\, v^{-1} \qquad (\eta_\ell = 1). \tag{1.6a}$$

Maximal inelasticity of the scattering in a given partial wave corresponds to $\eta_\ell = 0$. In the high energy limit this leads to the restriction

$$\mathrm{Im}\, f_\ell \le |f_\ell| \;\le\; 8\pi \qquad (\eta_\ell = 0). \tag{1.6b}$$

* Actual derivation of the unitarity condition for partial wave amplitudes uses the relation between the angles of initial, intermediate and final state particles and the known orthogonality properties of Legendre polynomials.

In this case the amplitude (1.5b) is purely imaginary, so that the elastic scattering is but a 'shadow' of inelastic channels. The model

$$f_\ell = \begin{cases} \mathrm{i}\,\dfrac{8\pi}{v}, & \eta_\ell = 0, \qquad\qquad \text{for } \ell < \ell_0 = k_s R, \\ 0\,, & \eta_\ell = 1, \delta_\ell = 0, \qquad \text{for } \ell > \ell_0, \end{cases}$$

is known as the 'black disk' model for diffractive scattering. At high energies $s \simeq 4k_s^2 \gg \mu^2$ ($v \simeq 1$) when $\ell_0 \gg 1$, it leads to the forward scattering amplitude (see (1.3a))

$$A(s,0) = \sum_\ell (2\ell+1) f_\ell \simeq \ell_0^2 \cdot 8\pi\mathrm{i} \simeq \mathrm{i}\, s \cdot 2\pi R^2,$$

which, according to the optical theorem, results in

$$\sigma_{\mathrm{tot}} = \frac{\mathrm{Im}\, A(s,0)}{v\, s} \simeq 2\pi R^2 = \pi R^2\Big|_{\mathrm{inelastic}} + \pi R^2\Big|_{\mathrm{diffraction}}.$$

This is the pattern of diffraction off an absorbing disk of radius R.

1.3.1 Threshold behaviour of partial wave amplitudes

It is well known from quantum mechanics that for potentials of finite range, r_0, the partial waves behave like $(kr_0)^\ell$ as $k \to 0$. It can be easily seen that a similar result holds in the theory of the S matrix.

Indeed, the singularity in t of the amplitude $A(s,t)$, the closest to the physical region in the s-channel, is located at $t = 4\mu^2$. Therefore the series (1.3a) should be convergent for z up to $z_0 = 1 + 4\mu^2/2k_s^2$.

For $t > 0$ and $s \to 4\mu^2$, one gets $z \to \infty$ and $P_\ell(z)$ grows as $P_\ell(z) \sim z^\ell$. For the series (1.3a) to converge, one has to require that f_ℓ should fall with ℓ like $\left(2k_s^2/4\mu^2\right)^\ell$ but not faster since at $t = 4\mu^2$ the series has to be divergent.

1.3.2 Singularities of $\mathrm{Im}\, A$ *on the Mandelstam plane (Karplus curve)*

Repeating the same arguments for the *imaginary part* of the s-channel amplitude $\mathrm{Im}\, A$ we would get

$$\mathrm{Im}\, f_\ell(s) \propto k_s^{2\ell}, \qquad k_s \to 0.$$

This cannot be true, however, since it contradicts the unitarity condition: $\mathrm{Im}\, f_\ell \propto k_s^{4\ell+1}$ follows from (1.4a). Substituting this behaviour into (1.3a), we observe that the series for $\mathrm{Im}_s A(s,t)$ remains convergent at $t = \mu^2$. We conclude that singularities in t of the *imaginary part* of the amplitude are located *above* $t = 4\mu^2$, and their position depends on s.

Actually, using the unitarity condition one can find the exact form of the line of singularities of $\mathrm{Im}_s A(s,t)$ on the Mandelstam plane, known as Karplus (or Landau) curve.

Let us sketch its derivation in the region $4\mu^2 \le s \le 16\mu^2$, $t > 4\mu^2$ where the two-particle unitarity condition is valid ($\Delta = 0$ in (1.4a)).

For $t > 0$ we have $z > 1$ and the Legendre polynomials increase exponentially with ℓ:

$$P_\ell(\cosh\alpha) \overset{\ell\to\infty}{\simeq} \frac{\mathrm{e}^{(\ell+\frac{1}{2})\alpha}}{\sqrt{2\pi\ell\sinh\alpha}}, \qquad \cosh\alpha \equiv z = 1 + \frac{t}{2k_s^2} > 1\,. \tag{1.7}$$

To ensure convergence of (1.3a) for $t < 4\mu^2$, partial waves have to fall as

$$f_\ell \sim \mathrm{e}^{-\ell\alpha_0}, \quad \cosh\alpha_0 = 1 + \frac{4\mu^2}{2k_s^2}. \tag{1.8}$$

Due to the unitarity condition (1.4a) the imaginary part falls even faster: $\mathrm{Im}\, f_\ell \sim \exp(-2\ell\alpha_0)$.

Consider now the series (1.3a) for $\mathrm{Im}_s A(s,t)$. With t increasing, the growing factor $\exp(\ell\alpha)$, originating from the Legendre polynomials, eventually overtakes the falling factor $\exp(-2\ell\alpha_0)$ due to $\mathrm{Im}\, f_\ell$. At this point the series becomes divergent, and $\mathrm{Im}_s A(s,t)$ develops a singularity.

Thus, the line of singularities of $\mathrm{Im}_s A(s,t)$ for $4\mu^2 \le s \le 16\mu^2$ is given by the equation $\alpha = 2\alpha_0$. In terms of the variables s and t this equation takes the form

$$\frac{t}{16\mu^2} = \frac{s}{s-4\mu^2}, \quad 4\mu^2 \le s \le 16\mu^2.$$

In the complementary region $4\mu^2 \le t \le 16\mu^2$, $s \ge 4\mu^2$, the Karplus curve can be found using the symmetry of $A(s,t)$ under the permutation $s \leftrightarrow t$:

$$\frac{s}{16\mu^2} = \frac{t}{t-4\mu^2}, \quad 4\mu^2 \le t \le 16\mu^2.$$

This example illustrates how the unitarity condition determines the analyticity domain of the scattering amplitude.

The lines of singularities C_i of the amplitude $A(s,t)$ are drawn on the Mandelstam plane in Fig. 1.2.

The fact that the Karplus curve $C_1(s,t)$ has finite asymptotes (in our example, the lines $s = 4\mu^2, t \to \infty$, and $t = 4\mu^2, s \to \infty$) is obvious, since otherwise the partial wave amplitudes would decrease with increasing ℓ faster than any exponential, which is in contradiction with the standard behaviour $f_\ell \sim \exp(-\alpha\ell)$ for $\ell \to \infty$.

In reality, the Karplus curves for $\pi\pi$ scattering are not symmetric with respect to s and t, which is a consequence of the pions being pseudoscalars (see the following lectures and the footnote on page 27).

1.4 The Froissart theorem

In 1958 Froissart showed that the analytic properties of the scattering amplitude together with the unitarity condition put certain restrictions on the asymptotic behaviour of $A(s,t)$ in the physical region. Let us show that asymptotically

$$\mathrm{Im}\, A(s,t)|_{t=0} \;\le\; \mathrm{const}\cdot s\ln^2\frac{s}{s_0}\,, \qquad s\to\infty.$$

First let us estimate f_ℓ at large s using the fact that the singularity of $\mathrm{Im}_{\,s}A(s,t)$ closest to the physical region of the s-channel is situated at $t=4\mu^2$. As was shown above, at large ℓ the partial wave amplitude falls exponentially. Since for $k_s^2 \propto s \gg t$ (1.8) gives $\alpha \simeq \sqrt{t}/k_s$, we have

$$f_\ell(s) \;\simeq\; c(s,\ell)\,\exp\left(-\frac{\ell}{k_s}\sqrt{4\mu^2}\right), \qquad \ell,\, s\to\infty, \tag{1.9}$$

where $c(s,\ell)$ is slowly (non-exponentially) varying with ℓ.

Let us now assume that for t arbitrarily close to $4\mu^2$ the amplitude grows with s not faster than some power. Then the same is valid for $\mathrm{Im}\; c(s,\ell)$. Indeed, $\mathrm{Im}\, f_l$ is positive due to the unitarity condition, and so is $P_\ell(1+t/2k_s^2)$ for $t\ge 0$. Therefore for each partial wave we have an estimate†

$$\begin{aligned}\left(\frac{s}{s_0}\right)^N &> \mathrm{Im}\, A(s,t) = \sum_{\ell=0}^{\infty}\mathrm{Im}\, f_\ell(s)(2\ell+1)P_\ell\left(1+\frac{t}{2k_s^2}\right)\\ &> \mathrm{Im}\; c(s,\ell)\left(2\pi\ell\frac{\sqrt{t}}{k_s}\right)^{-1/2}\exp\left\{\frac{\ell}{k_s}\left(\sqrt{t}-\sqrt{4\mu^2}\right)\right\}.\end{aligned} \tag{1.10}$$

Since (1.10) holds for arbitrary positive $t<4\mu^2$, we conclude that

$$\mathrm{Im}\; c(s,\ell) < (s/s_0)^N\,,$$

and finally, modulo an irrelevant pre-exponential factor,

$$\mathrm{Im}\, f_\ell(s) \;\lesssim\; \left(\frac{s}{s_0}\right)^N \exp\left(-\frac{2\mu}{k_s}\ell\right). \tag{1.11}$$

(Using the unitarity condition one can derive a similar estimate for $\mathrm{Re}\, f_\ell$.)

† the series converges inside the so-called Lehman ellipse in the z plane

We are now in a position to estimate the imaginary part of the forward scattering amplitude:

$$\begin{aligned}\operatorname{Im} A(s,t=0) &= \sum_{\ell=0}^{\infty} \operatorname{Im} f_\ell(s)\,(2\ell+1) \\ &\le 8\pi \sum_{\ell=0}^{L}(2\ell+1) + \sum_{\ell=L+1}^{\infty} \operatorname{Im} f_\ell(s)(2\ell+1). \qquad (1.12)\end{aligned}$$

Here we have extracted the finite sum $\ell < L$ in which partial waves are large, $\operatorname{Im} f_\ell \simeq |f_\ell| = \mathcal{O}(1)$, and estimated its contribution from above by substituting for $\operatorname{Im} f_\ell$ its maximal value allowed by unitarity, see (1.6b):

$$\sum_{\ell=0}^{L}(2\ell+1) \;\simeq\; L^2\,.$$

The border value of the angular momentum L above which partial wave amplitudes become small, $\operatorname{Im} f_{\ell>L} \ll 1$, and fall exponentially with ℓ according to (1.11) can be found by setting

$$\left(\frac{s}{s_0}\right)^N \exp\left(-\frac{2\mu}{k_s}\,L\right) \simeq 1 \qquad \Longrightarrow \qquad L \simeq \frac{k_s}{2\mu}\,\ln\frac{s}{s_0}\,.$$

The contribution of the infinite tail of the series in (1.12) can be estimated using $f_{L+n} \sim f_L \exp(-2\mu\, n/k_s)$ and turns out to be subdominant:

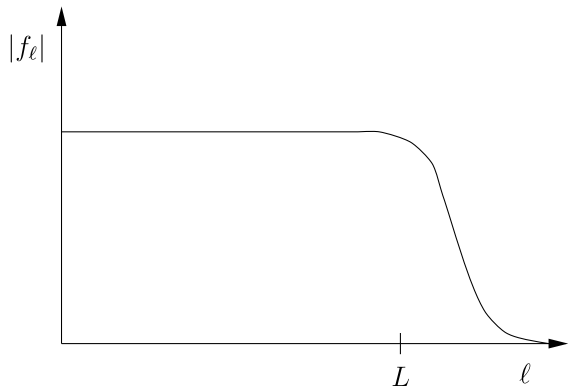
$$\sum_{n=0}^{\infty} 2(L+n)\,\exp\left\{-\frac{2\mu}{k_s} n\right\} \simeq \frac{k_s}{\mu}\,L + \frac{k_s^2}{2\mu^2} \overset{s\to\infty}{\ll} L^2.$$

Thus,

$$\operatorname{Im} A(s,t=0) \;\propto\; L^2 \;\propto\; s\ln^2\frac{s}{s_0}\,.$$

This is the Froissart theorem.

The magnitude of the partial wave as a function of ℓ is sketched here:



Since according to the optical theorem $\operatorname{Im} A(s, t = 0) = s\sigma_{\text{tot}}(s)$, it follows from the Froissart theorem that the total cross section cannot grow with the centre of mass energy $\sqrt{s}$ faster than the squared logarithm of s, $\sigma_{\text{tot}}(s) \le \sigma_0 \ln^2(s/s_0)$, and the interaction radius cannot grow faster than the logarithm of s.

An analogous consideration, together with the unitarity condition, leads to the similar inequality for the *real* part of the forward scattering amplitude, $|\operatorname{Re} A(s, t = 0)| < \text{const} \cdot s \ln^2(s/s_0)$.

In order for the cross section not to decrease with increasing energy, the amplitude $A(s, t = 0)$ has to grow and, as a consequence, the number of partial waves contributing to the sum in (1.3a) has to be large. This allows us to replace the sum in (1.3a) by the integral over ℓ, using the well known approximate expression for the Legendre polynomials,

$$P_\ell(\cos\Theta) \simeq J_0\left[(2\ell+1)\frac{\Theta}{2}\right], \qquad \ell \gg 1, \quad \theta \ll 1. \tag{1.13}$$

We obtain

$$A(s,t) \simeq \int f_\ell(s)\, J_0\left[(2\ell+1)\frac{\Theta}{2}\right](2\ell+1)\mathrm{d}\ell.$$

It is convenient to replace ℓ by the impact parameter ρ, $\ell + 1/2 = k_s\rho$. Then, using $t \simeq -(k_s\Theta)^2$, we obtain

$$A(s,t) \simeq k_s^2 \int f(\rho, s)\, J_0\left(\rho\sqrt{-t}\right) 2\rho\, \mathrm{d}\rho. \tag{1.14}$$

If the values of ρ giving the dominant contribution to this integral do not depend on s (which is the case for the usual picture of diffractive scattering off a finite size object), then it is natural to expect that the amplitude takes the factorized form $A(s,t) \simeq a(s)F(t)$. If we additionally assume that the partial wave amplitudes $f(\rho, s)$ that are dominant in (1.11) approach constant values as $s \to \infty$, then $A(s,t) \sim sF(t)$ and the total cross section tends to a constant.

1.5 The Pomeranchuk theorem

In 1958 I.Ya. Pomeranchuk showed that if the total cross sections are constant at high energies, then the total cross sections of the scattering of a particle and its antiparticle off the same target should be asymptotically equal. The derivation of this result is based on the properties of the scattering amplitude in the s- and u-channels.

Let us identify the singularities of $A(s, t = 0)$ in the complex s plane. They are the right-hand cut $s \ge 4\mu^2$ and the left-hand cut $s \le 0$. The latter cut corresponds to the right-hand cut $u \ge 4\mu^2$ in the complex u plane due to the relation $s + t + u = 4\mu^2$.

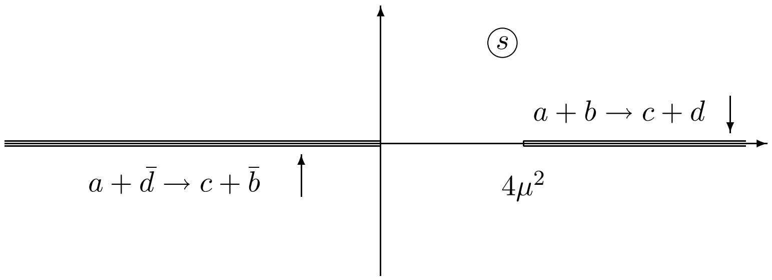


Fig. 1.3. Amplitudes of two crossing reactions in the complex s plane

It is natural to assume that the amplitude of the reaction $a+b \to c+d$ is equal to the value $A(s,t)$ on the upper edge of the right cut in s, which corresponds to the usual definition of Feynman integrals in perturbation theory:

$$A(a+b \to c+d) \to \lim_{\varepsilon \to 0} A(s + \mathrm{i}\,\varepsilon, t).$$

Similarly, the physical amplitude of the reaction $a+\bar{d} \to c+\bar{b}$ is given by the value of A on the upper edge of the right-hand cut in u, i.e.

$$A(a+\bar{d} \to c+\bar{b}) = \lim_{\varepsilon \to 0} A(u + \mathrm{i}\,\varepsilon, t) = \lim_{\varepsilon \to 0} A(-(s - \mathrm{i}\,\varepsilon) - t + 4\mu^2, t),$$

where the latter equality follows from the identity $s+t+u = 4\mu^2$ together with crossing symmetry. Thus the physical amplitude of the cross-channel reaction in the s plane is obtained by approaching the cut $s \le 0$ *from below*, as shown in Fig. 1.3. Furthermore, since $A(s, t \le 0)$ is real on the interval $0 < s < 4\mu^2$ which is free from singularities, the values of the amplitude on the two edges of the cut are complex conjugate. Therefore we may use the relation $A(s - \mathrm{i}\,\varepsilon, t < 0) = A^*(s + \mathrm{i}\,\varepsilon, t < 0)$ to finally arrive at

$$A_{a+\bar{d}\to c+\bar{b}}(s) \;\simeq\; \left[\, A_{a+b\to c+d}(-s)\,\right]^*, \qquad s \simeq -u\,. \tag{1.15}$$

Pomeranchuk proved the theorem under the assumption that the elastic scattering amplitude at large s has the form

$$A_{a+b\to a+b} \;=\; s\,F(t)\,, \tag{1.16a}$$

so that the total cross section tends to a constant at $s \to \infty$. Using the relation (1.15) we then obtain

$$A_{a+\bar{b}\to a+\bar{b}} \;=\; -s\,F^*(t)\,, \tag{1.16b}$$

yielding that the imaginary parts of the two amplitudes are equal whereas their real parts have opposite signs. (This implies that in such a model the part of the amplitude that is symmetric in s, u must be purely imaginary while the antisymmetric part must be real.) Since the total cross section is defined by the imaginary part of A, the Pomeranchuk theorem follows suit:

$$\sigma_{\rm tot}(a+b) \;=\; \sigma_{\rm tot}(a+\bar{b}).$$

If the total cross sections *increase* with energy, the asymptotic equality of σ_{ab} and $\sigma_{a\bar{b}}$ cannot, in general, be proved. The Pomeranchuk theorem, however, can be proved, assuming asymptotic factorization of the amplitude, $A(s,t) \simeq a(s)F(t)$, for a special class of the energy behaviour, namely, $a(s) = s(\ln s)^{\beta}$. To carry out the proof one must use the hypothesis that asymptotically the real part of the amplitude does not exceed its imaginary part

$$\lim_{s\to\infty} \frac{\mathrm{Re}\, A(s,t)}{\mathrm{Im}\, A(s,t)} \;<\; \mathrm{const}. \tag{1.17}$$

It is supported by the observation that in general $\mathrm{Re}\, f_\ell$ is a sign alternating function so that destructive interference in the series (1.3a) for $\mathrm{Re}\, A(s,t)$ is possible. (Here it is important, once again, that at high energies the large values of ℓ are essential.)

We may illustrate the nature and significance of this hypothesis on a simple example. Consider an amplitude of the form

$$A(s,t) = s\ln\frac{-s}{s_0}\cdot c(t)$$

with $c(t)$ a real function. For $s>0$ this amplitude is complex, and the cross section in the s-channel is constant, whereas at negative s (positive u) we have $\mathrm{Im}\, A = 0$ and the u-channel cross section vanishes.

Did we manage to construct a counterexample to the Pomeranchuk theorem? Obviously not, since our model amplitude is not realistic. It gives rise to the elastic cross section exceeding the total cross section,

$$\sigma_{\rm el} \sim \int \frac{\mathrm{d}t}{s^2}\,|A(s,t)|^2 \propto \ln^2 s \;\gg\; \sigma_{\rm tot} \sim \mathrm{const},$$

which is a consequence of $\mathrm{Re}\, A/\,\mathrm{Im}\, A \sim \ln s \to \infty$, in contradiction with (1.17).

In this lecture we have demonstrated simple consequences of the analyticity and crossing symmetry of the scattering amplitude.

In the forthcoming lectures we will show how the t-channel unitarity can be used to study the asymptotics of the scattering amplitudes for $s\to\infty$. It is singularities of the amplitude in t (rather than those in u) that are located close to the physical region in the s-channel on the Mandelstam plane. This explains why the physics of the t-channel is important for large s.