Introduction to Symmetry Analysis

BRIAN J. CANTWELL
Stanford University

CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First published 2002

Printed in the United Kingdom at the University Press, Cambridge

This book was set using LaTeX and Mathematica™

Typeface Times Roman 10/13 pt. *System* LaTeX 2_ε [TB]

A catalog record for this book is available from the British Library.

Library of Congress Cataloging in Publication Data
Cantwell, Brian.
Introduction to symmetry analysis / Brian J. Cantwell.
p. cm. – (Cambridge texts in applied mathematics)
Includes bibliographical references and index.
ISBN 0-521-77183-8 – ISBN 0-521-77740-2
1. Differential equations – Numerical solutions 2. Symmetry (Physics)
I. Title. II. Series.
QA371 .C195 2002
515′.35 – dc21 2001037835

ISBN 0 521 77183 8 hardback
ISBN 0 521 77740 2 paperback

Contents

1

Introduction to Symmetry

1.1 Symmetry in Nature

Symmetry is universal, fascinating, and of immense practical importance. As human beings we have evolved a perception of symmetry that lies at the core of our conscious life. Symmetries provide cues that help us relate to our environment and guide our movements through the world. Everyone has a taste for things that are in some way symmetrical or possess a pleasing deviation from perfect symmetry. A highly paid supermodel will often have rather symmetrical facial features. But a perfectly symmetrical face has an unnatural, androgynous look, and rarely is this associated with great beauty or a memorable persona. Perhaps the most perfect object we can imagine is a circle, yet dividing the circumference by the diameter produces the irrational number π that we can only symbolize. Perfect, unequivocal, symmetry, like perfect theory, eludes us always.

Objects of the natural world universally exhibit some form of symmetry. Despite an astonishing variety of shapes, all members of the animal kingdom possess body architectures that can be sorted into only about 37 basic types. Almost all animals possess bilateral symmetry; they must eat, and to eat efficiently two hands, grasping symmetrically, are better than one. Animals must move, and to move efficiently it is essential to be balanced about the center of mass. When asymmetric development does occur, it is invaribly associated with some unusual, very specific adaptation, as in the case of the bottom-dwelling flounder with both eyes on the same side of its head. The whorls and spirals of plant organs produced by the response of an expanding growth surface to surrounding mechanical constraints [1.1] have been the subject of scientific inquiry for centuries. The nearly perfect spheres that fill the universe – stars, planets, moons, and the like – are shaped primarily by gravitational forces, which act in a three-dimensional universe where no one direction or position is distinguished from another. Free space is homogeneous and isotropic. We marvel at the incredible variety of delicate geometrical forms associated with the six-sided symmetry of snowflakes or the regular crystalline structure of gems formed over millennia by heat, pressure, and water, their shape a consequence of the forces that act on an atomic scale according to the symmetries of the electronic outer shells that participate in bonding. Anyone who studies fluid mechanics is struck by the aesthetic symmetry of shock wave patterns or bubbly flows or any of the myriad spiral patterns that mark the vortical world that flows over, around, and through us.

There have been many attempts to quantify the relationship between symmetry and beauty. A fine example of this can be found in the fascinating work of George David Birkhoff (1884–1944) [1.2], who was one of the preeminent American mathematicians of the early 20th century and is generally credited with developing the ergodic theorem in the kinetic theory of gases. Birkhoff was originally motivated by the desire to identify what it was that made one musical piece beautiful and another not. He felt that beauty had a universal character and therefore it should be possible to quantify it mathematically, and so he developed what he called the "aesthetic measure." Ultimately he applied this measure to a wide variety of objects – everything from musical pieces to vases to floor tilings. Today such an attempt to categorize music seems naive in view of the vast range of musical technique – everything from guitar "resonant buzz" invented accidentally by country singer Marty Robbins (but claimed by "Spirit in the Sky" Norman Greenbaum) to the patriotic screechings of Jimi Hendrix to the asynchronous beat of Dave Brubeck. No simple measure can cover it all.

Although the use of symmetries to categorize objects is interesting in its own right, that is not the purpose of this text. Our main interest is in the symmetries inherent in the physical laws that govern the natural world. Knowledge of these symmetries will be used to enhance our understanding of complex physical phenomena, to simplify and solve problems, and, ultimately, to deepen our understanding of nature. The primary goal of this text is to develop the methods of symmetry analysis based on Lie groups for the uninitiated reader and to use these methods to find and express the symmetry properties of ordinary differential equations, partial differential equations, integrals, and the solution functions that they govern. The text is directed primarily at first- and second-year graduate students in science and engineering, but it may also be useful to advanced researchers who would like to gain some familiarity with symmetry methods. The student is expected to be familiar with classical approaches to the solution of differential equations, although the early chapters provide much of the required background in terms that should be understandable to an upper-level undergraduate.

1.2 Some Background

My first encounter with Lie groups came while browsing in the GALCIT aeronautics library at Caltech in 1975. I ran across the book by Abraham Cohen [1.3], first published in 1911. The first few chapters of this book give a very lucid description of the concept of a Lie group and the idea of invariance under a group. Cohen's book makes interesting reading when one realizes that at the time it was written, Sophus Lie's ideas were still a brand-new development, yet they were seen as important enough to warrant a full-blown textbook treatment. In his 1906 treatise on *The Theory of Differential Equations* Andrew Forsyth devotes several chapters to Lie groups and Bäcklund transformations. It is a fact, however, that shortly thereafter, Lie's ideas fell into obscurity and remained so until soon after World War II. As researchers began to turn more and more often to nonlinear problems and as the inherent importance of symmetries began to be recognized, Lie's ideas gained renewed interest.

The Lie algorithm used to analyze the symmetry of mathematical expressions was developed to an advanced state through the pioneering efforts of Ovsiannikov [1.5] and his students in the Soviet Union. In the United States, Garrett Birkhoff [1.6] at Harvard the son of George Birkhoff played a key role in bringing attention to Lie's ideas by clarifying the relationship between group invariance and dimensional analysis as applied to problems in fluid mechanics. Fluid mechanics, governed as it is by nonlinear equations from which a rich variety of simplified nonlinear and linear approximations can be derived, is an especially fertile source of examples and applications of group theory.

During the same period, new ideas about the role of similarity solutions as approximations to realistic complex physical problems were being developed by Barenblatt and Zel'dovich [1.7] in the Soviet Union. By the late 1960s and early 1970s the whole field was active again, and new applications of group theory were being developed by a number of researchers, including Ibragimov in the Soviet Union [1.8], Bluman and Cole at Caltech [1.9], Anderson, Kumei, and Wulfman at the University of the Pacific [1.10], Chester at Bristol [1.11], Harrison and Estabrook at the Jet Propulsion Laboratory [1.12], and many others. Today group analysis, in one form or another, is the central topic of a number of excellent textbooks, including Hansen [1.13], Ames [1.14], Olver [1.15], Bluman and Kumei [1.16], Rogers and Ames [1.17], Stephani [1.18], and most recently Ibragimov [1.19], Andreev et al. [1.20], Hydon [1.21] and Baumann [1.22]. The valuable collection of results by workers around the world contained in the CRC series edited by Ibragimov [1.23] gives testimony to the achievements of the last half century or so. Today, symmetry analysis constitutes the most important (indeed one might say the only) widely applicable method for finding analytical solutions of nonlinear problems. The Lie algorithm can be applied to virtually any system of ODEs and PDEs. Moreover the procedure is highly systematic and amenable to programming with symbol manipulation software. As a result, sophisticated software tools are now available for analyzing the symmetries of differential equations (References [1.24], [1.25], [1.26]; see also the review of symbolic software for group analysis by Hydon [1.21] and Hereman [1.27]).

1.3 The Discrete Symmetries of Objects

For more background on the importance of symmetry, particularly in the early development of modern physics, I would recommend the works of the German–American mathematical physicist Hermann Weyl (1885–1955), who formulated the group-theoretic basis of quantum mechanics. In his monograph [1.28] Weyl writes of the role of symmetry in science and art. Weyl was a student of David Hilbert and a member of the famous group of German mathematicians at the University of Göttingen, which broke up during the Nazi era prior to the start of World War II and later re-formed as the nucleus of the Courant Institute in New York. Finally, one of my favorite readings is Feynman's discussion of the role of symmetry in modern physics, which can be found in Chapter 52 of Volume I of the *Feynman Lectures on Physics* [1.29].

Let's begin with a widely accepted general definition of symmetry usually attributed to Weyl.

Definition 1.1. *An object is symmetrical if one can subject it to a certain operation and it appears exactly the same after the operation. The object is then said to be invariant with respect to the given operation.*

The symmetry properties of an object can usually be expressed in terms of a set of matrices each of which, when used to transform the various points composing the object, leave it unchanged in appearance. To clarify the notion of symmetry and its mathematical description, let's examine the rotational and reflectional symmetry of a snowflake.

1.3.1 The Twelvefold Discrete Symmetry Group of a Snowflake

Transparent ice crystals form around dust particles in the atmosphere when water vapor condenses at temperatures below the freezing point. The water molecule is an isosceles triangle composed of two hydrogen atoms bonded to an oxygen atom at its apex with an angle of 104.5° between the bonds. The attraction between the hydrogen atoms of each molecule and the oxygen atoms of other molecules overcomes thermal motions, leading to the formation of

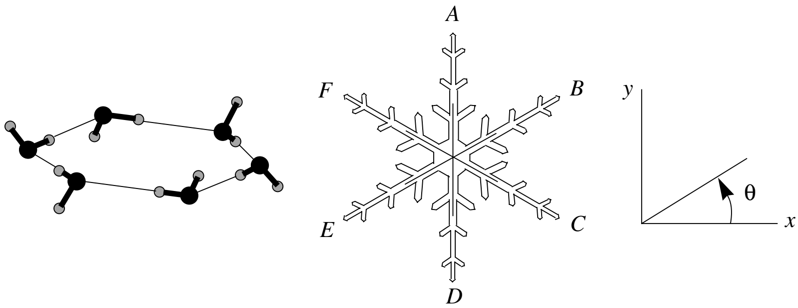

Fig. 1.1. Hexagonal structure of ice crystals and snowflakes.

hydrogen bonds, which link molecules together. The symmetry properties of the water molecule are such that if the formation temperature is below $-14°$C, each molecule bonds to four neighboring molecules in a repeating tetrahedral arrangement with the oxygen atoms at the corners of the tetrahedron. The tetrahedral structure gives rise to hexagonal rings of water molecules as shown in Figure 1.1. These hexagons on the molecular scale are responsible for the hexagonal symmetry of the ice crystal at macroscopic scales.

The exact structure of the ice crystal depends on its temperature history during formation. Thus, because of the infinite variability of atmospheric conditions, the shape of each snowflake is unique.

One final point before we begin: A snowflake is a three-dimensional object with a front and back. Here we wish to study only the planar symmetry of a face-on view, and so we consider the snowflake to be flat, existing entirely in a two-dimensional world. By the way, the tendency for snowflakes to be nearly flat is also explained by the crystal structure at the molecular level, which tends to be composed of relatively weakly bound planar sheets.

Figure 1.1 is my best attempt to sketch a typical snowflake. Overall it looks like a fairly symmetrical six-sided object. However, close inspection reveals a lot of detailed imperfections in my drawing. In order to have a useful discussion of the symmetry properties of the snowflake, we simply must accept the fact that we can't look at it too closely. We have to be willing to gloss over the imperfections and agree that the six corners of the snowflake are indistinguishable. The labels A, B, C, D, E, F are applied to the corners for reference purposes, but with the convention that the labels do not compromise the property that the corners themselves are indistinguishable.

This seemingly minor point is actually crucial and all-encompassing. It is central to the methods used to test for symmetry. In principle, any real object in all of its detail is completely devoid of symmetry. Therefore it is important to recognize that the symmetries that accrue to an object apply, not to the object itself, but to its abstract representation. The moon is a sphere only when viewed from a perspective that flattens all mountain ranges, mare, rocks, pebbles, etc. Often it is the degree and manner in which a symmetry is broken that is of paramount importance. Galileo's great discovery in the seventeenth century was that the moon is not a smooth sphere but is covered with craters whose dimensions rival the largest geological features found on earth.

So it is the case today that the most important scientific questions are often associated with peeling away symmetries or searching for new symmetries of complex systems in order to reach a deeper understanding of the underlying physics. One often asks: Which parameters in a physical problem are important? Which ones are not? Occasionally, new physics is discovered when the means is found to "fix" a broken symmetry. In the modern era, the most spectacular example of this is the failure of Maxwell's equations to preserve Galilean invariance while preserving invariance under the puzzling Lorentz transformation. This led directly to Einstein's theory of special relativity, the recognition that time and space are connected, and the discovery that the speed of light is a universal invariant for all observers. A more recent example that shook the foundations of particle physics is the famous 1956 discovery by Lee and Yang [1.30], [1.31] that parity is not conserved in beta decay.

1.3.1.1 Symmetry Operations

Now, let's begin our study of the symmetries of a snowflake.

Suppose we rotate the snowflake by 30° (Figure 1.2). If we close our eyes before the rotation, then open them afterwards, we can see that an operation has been applied to the snowflake. The object is not left invariant, and the 30° rotation does not qualify as a symmetry operation. There are in fact just six rotation angles that leave the snowflake invariant: 60°, 120°, 180°, 240°, 300°, and 360°.

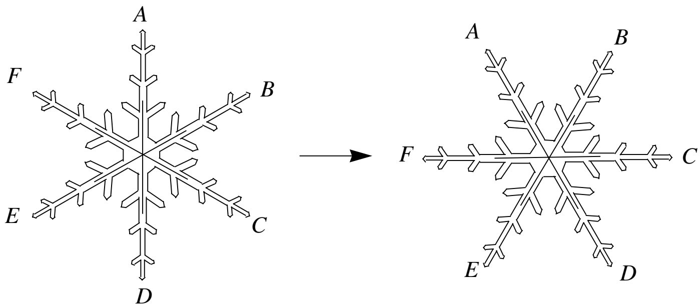

Fig. 1.2. Counterclockwise rotation by 30°.

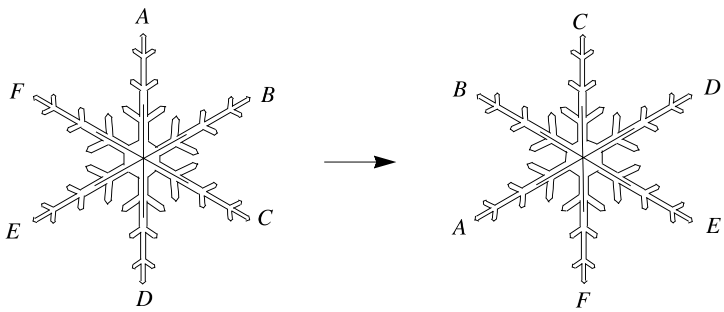

Fig. 1.3. Counterclockwise rotation by 120°.

Now apply a rotation of 120° (Figure 1.3). In this case, there is no way we can tell that the operation has taken place (remember that the labels are not part of the object and tiny details are ignored). The snowflake is invariant, and the rotation by 120° is a symmetry operation. We can express the rotational symmetry of the snowflake mathematically as a transformation

$$\begin{aligned} \tilde{x} &= x\cos\theta - y\sin\theta, \\ \tilde{y} &= x\sin\theta + y\cos\theta. \end{aligned} \tag{1.1}$$

where the (x, y) coordinates are oriented as shown in Figure 1.1 and the parameter of the transformation, θ, can only take on the six discrete values given above. It is convenient (though not necessary) to think of (1.1) as a mapping of points in a given space whose coordinate axes remain fixed, rather than the usual interpretation as a rotation of the coordinate axes themselves. The object moves under the action of the transformation while the reference axes stay fixed. The six rotations are as follows:

$$C_6^1 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \quad C_6^2 = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}, \quad C_6^3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$C_6^4 = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}, \quad C_6^5 = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{1.2}$$

The matrices $C_6^1,\ C_6^2,\ C_6^3,\ C_6^4,\ C_6^5, E$ express the rotational symmetry of *any* hexagonal object with indistinguishable sides and corners.

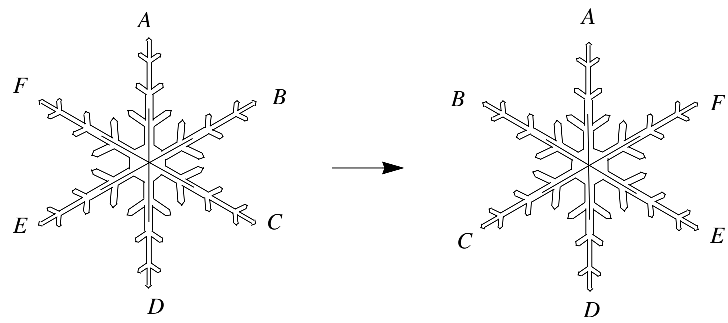

Fig. 1.4. Reflection through a vertical axis.

What about reflections? Reflection through an axis passing through A–D leaves the snowflake invariant (Figure 1.4). Recall that we are considering a flat snowflake and so all operations are in the plane of the paper. If we wanted to consider the three-dimensional symmetries of a finite-thickness snowflake, then we would have to include transformations in the z-direction, either reflecting points between the front and back or rotating the object out of the plane of the paper.

The reflection through A–D can be expressed as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}. \tag{1.3}$$

Another reflectional symmetry is through axis a–d, which splits the angle between A–D and B–E as shown in Figure 1.5. Four other symmetry operations are: reflection through axis B–E, reflection through C–F and reflections

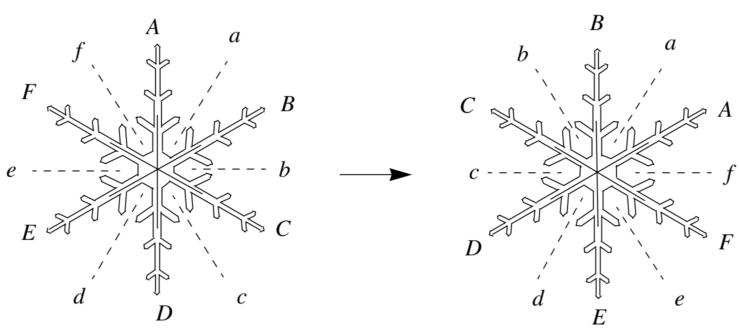

Fig. 1.5. Reflection axes of a snowflake.