10

Type I and type II superstrings

Having spent volume one on a thorough development of the bosonic string, we now come to our real interest, the supersymmetric string theories. This requires a generalization of the earlier framework, enlarging the world-sheet constraint algebra. This idea arises naturally if we try to include spacetime fermions in the spectrum, and by guesswork we are led to *superconformal symmetry*. In this chapter we discuss the (1,1) superconformal algebra and the associated type I and II superstrings. Much of the structure is directly parallel to that of the bosonic string so we can proceed rather quickly, focusing on the new features.

10.1 The superconformal algebra

In bosonic string theory, the mass-shell condition

$$p_\mu p^\mu + m^2 = 0 \tag{10.1.1}$$

came from the physical state condition

$$L_0|\psi\rangle = 0\ , \tag{10.1.2}$$

and also from $\tilde{L}_0|\psi\rangle = 0$ in the closed string. The mass-shell condition is the Klein–Gordon equation in momentum space. To get spacetime fermions, it seems that we need the Dirac equation

$$ip_\mu \Gamma^\mu + m = 0 \tag{10.1.3}$$

instead. This is one way to motivate the following generalization, and it will lead us to all the known consistent string theories.

Let us try to follow the pattern of the bosonic string, where L_0 and $\tilde{L}_0$ are the center-of-mass modes of the world-sheet energy-momentum tensor $(T_B, \tilde{T}_B)$. A subscript B for 'bosonic' has been added to distinguish these from the fermionic currents now to be introduced. It seems then that we need new conserved quantities T_F and $\tilde{T}_F$, whose center-of-mass modes give the Dirac equation, and which play the same role as T_B and $\tilde{T}_B$ in the bosonic theory. Noting further that the spacetime momenta p^μ are the center-of-mass modes of the world-sheet current $(\partial X^\mu, \bar{\partial} X^\mu)$, it is natural to guess that the gamma matrices, with algebra

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} , \tag{10.1.4}$$

are the center-of-mass modes of an anticommuting world-sheet field ψ^μ.

With this in mind, we consider the world-sheet action

$$S = \frac{1}{4\pi} \int d^2z \left(\frac{2}{\alpha'} \partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial} \psi_\mu + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu \right) . \tag{10.1.5}$$

For reference we recall from chapter 2 the XX operator product expansion (OPE)

$$X^\mu(z,\bar{z}) X^\nu(0,0) \sim -\frac{\alpha'}{2} \eta^{\mu\nu} \ln |z|^2 . \tag{10.1.6}$$

The ψ conformal field theory (CFT) was described in section 2.5. The fields ψ^μ and $\tilde{\psi}^\mu$ are respectively holomorphic and antiholomorphic, and the operator products are

$$\psi^\mu(z)\psi^\nu(0) \sim \frac{\eta^{\mu\nu}}{z} , \quad \tilde{\psi}^\mu(\bar{z})\tilde{\psi}^\nu(0) \sim \frac{\eta^{\mu\nu}}{\bar{z}} . \tag{10.1.7}$$

The *world-sheet supercurrents*

$$T_F(z) = i(2/\alpha')^{1/2} \psi^\mu(z) \partial X_\mu(z) , \quad \tilde{T}_F(\bar{z}) = i(2/\alpha')^{1/2} \tilde{\psi}^\mu(\bar{z}) \bar{\partial} X_\mu(\bar{z}) \tag{10.1.8}$$

are also respectively holomorphic and antiholomorphic, since they are just the products of (anti)holomorphic fields. The annoying factors of $(2/\alpha')^{1/2}$ could be eliminated by working in units where $\alpha' = 2$, and then be restored if needed by dimensional analysis. Also, throughout this volume the : : normal ordering of coincident operators will be implicit.

This gives the desired result: the modes ψ^μ_0 and $\tilde{\psi}^\mu_0$ will satisfy the gamma matrix algebra, and the centers-of-mass of T_F and $\tilde{T}_F$ will have the form of Dirac operators. We will see that the resulting string theory has spacetime fermions as well as bosons, and that the tachyon is gone.

From the OPE and the Ward identity it follows (exercise 10.1) that the currents

$$j^\eta(z) = \eta(z) T_F(z) , \quad \tilde{\jmath}^\eta(\bar{z}) = \eta(z)^* \tilde{T}_F(\bar{z}) \tag{10.1.9}$$

generate the *superconformal transformation*

$$\epsilon^{-1}(2/\alpha')^{1/2} \delta X^\mu(z,\bar{z}) = +\, \eta(z)\psi^\mu(z) + \eta(z)^* \tilde{\psi}^\mu(\bar{z}) , \tag{10.1.10a}$$

$$\epsilon^{-1}(\alpha'/2)^{1/2} \delta \psi^\mu(z) = -\, \eta(z) \partial X^\mu(z) , \tag{10.1.10b}$$

$$\epsilon^{-1}(\alpha'/2)^{1/2} \delta \tilde{\psi}^\mu(\bar{z}) = -\, \eta(z)^* \bar{\partial} X^\mu(\bar{z}) . \tag{10.1.10c}$$

This transformation mixes the commuting field X^μ with the anticommuting fields ψ^μ and $\tilde{\psi}^\mu$, so the parameter $\eta(z)$ must be anticommuting. As with conformal symmetry, the parameters are arbitrary holomorphic or antiholomorphic functions. That this is a symmetry of the action (10.1.5) follows at once because the current is (anti)holomorphic, and so conserved.

The commutator of two superconformal transformations is a conformal transformation,

$$\delta_{\eta_1}\delta_{\eta_2} - \delta_{\eta_2}\delta_{\eta_1} = \delta_v \ , \quad v(z) = -2\eta_1(z)\eta_2(z) \ , \tag{10.1.11}$$

as the reader can check by acting on the various fields. Similarly, the commutator of a conformal and superconformal transformation is a superconformal transformation. The conformal and superconformal transformations thus close to form the *superconformal algebra*. In terms of the currents, this means that the OPEs of T_F with itself and with

$$T_B = -\frac{1}{\alpha'}\partial X^\mu \partial X_\mu - \frac{1}{2}\psi^\mu \partial \psi_\mu \tag{10.1.12}$$

close. That is, only T_B and T_F appear in the singular terms:

$$T_B(z)T_B(0) \sim \frac{3D}{4z^4} + \frac{2}{z^2}T_B(0) + \frac{1}{z}\partial T_B(0) \ , \tag{10.1.13a}$$

$$T_B(z)T_F(0) \sim \frac{3}{2z^2}T_F(0) + \frac{1}{z}\partial T_F(0) \ , \tag{10.1.13b}$$

$$T_F(z)T_F(0) \sim \frac{D}{z^3} + \frac{2}{z}T_B(0) \ , \tag{10.1.13c}$$

and similarly for the antiholomorphic currents. The $T_B T_F$ OPE implies that T_F is a tensor of weight $(\frac{3}{2}, 0)$. Each scalar contributes 1 to the central charge and each fermion $\frac{1}{2}$, for a total

$$c = (1 + \tfrac{1}{2})D = \tfrac{3}{2}D \ . \tag{10.1.14}$$

This enlarged algebra with T_F and $\tilde{T}_F$ as well as T_B and $\tilde{T}_B$ will play the same role that the conformal algebra did in the bosonic string. That is, we will impose it on the states as a constraint algebra — it must annihilate physical states, either in the sense of old covariant quantization (OCQ) or of Becchi–Rouet–Stora–Tyutin (BRST) quantization. Because of the Minkowski signature of spacetime the timelike ψ^0 and $\tilde{\psi}^0$, like X^0, have opposite sign commutators and lead to negative norm states. The fermionic constraints T_F and $\tilde{T}_F$ will remove these states from the spectrum.

More generally, the $N = 1$ superconformal algebra in operator product

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form is

$$T_B(z)T_B(0) \sim \frac{c}{2z^4} + \frac{2}{z^2}T_B(0) + \frac{1}{z}\partial T_B(0) \,, \tag{10.1.15a}$$

$$T_B(z)T_F(0) \sim \frac{3}{2z^2}T_F(0) + \frac{1}{z}\partial T_F(0) \,, \tag{10.1.15b}$$

$$T_F(z)T_F(0) \sim \frac{2c}{3z^3} + \frac{2}{z}T_B(0) \,. \tag{10.1.15c}$$

The Jacobi identity requires the same constant c in the $T_B T_B$ and $T_F T_F$ products (exercise 10.5). Here, $N = 1$ refers to the number of $(\frac{3}{2}, 0)$ currents. In the present case there is also an antiholomorphic copy of the same algebra, so we have an $(N, \tilde{N}) = (1, 1)$ *superconformal field theory (SCFT)*. We will consider more general algebras in section 11.1.

Free SCFTs

The various free CFTs described in chapter 2 have superconformal generalizations. One free SCFT combines an anticommuting bc theory with a commuting $\beta\gamma$ system, with weights

$$h_b = \lambda \,, \quad h_c = 1 - \lambda \,, \tag{10.1.16a}$$

$$h_\beta = \lambda - \tfrac{1}{2} \,, \quad h_\gamma = \tfrac{3}{2} - \lambda \,. \tag{10.1.16b}$$

The action is

$$S_{BC} = \frac{1}{2\pi}\int d^2z\,(b\bar{\partial}c + \beta\bar{\partial}\gamma) \,, \tag{10.1.17}$$

and

$$T_B = (\partial b)c - \lambda\partial(bc) + (\partial\beta)\gamma - \frac{1}{2}(2\lambda - 1)\partial(\beta\gamma) \,, \tag{10.1.18a}$$

$$T_F = -\frac{1}{2}(\partial\beta)c + \frac{2\lambda - 1}{2}\partial(\beta c) - 2b\gamma \,. \tag{10.1.18b}$$

The central charge is

$$[-3(2\lambda - 1)^2 + 1] + [3(2\lambda - 2)^2 - 1] = 9 - 12\lambda \,. \tag{10.1.19}$$

Of course there is a corresponding antiholomorphic theory.

We can anticipate that the superconformal ghosts will be of this form with $\lambda = 2$, the anticommuting $(2, 0)$ ghost b being associated with the commuting $(2, 0)$ constraint T_B as in the bosonic theory, and the commuting $(\frac{3}{2}, 0)$ ghost β being associated with the anticommuting $(\frac{3}{2}, 0)$ constraint T_F. The ghost central charge is then $-26 + 11 = -15$, and the condition that the total central charge vanish gives the critical dimension

$$0 = \frac{3}{2}D - 15 \Rightarrow D = 10 \,. \tag{10.1.20}$$

For $\lambda = 2$,

$$T_B = -(\partial b)c - 2b\partial c - \frac{1}{2}(\partial\beta)\gamma - \frac{3}{2}\beta\partial\gamma \ , \tag{10.1.21a}$$

$$T_F = (\partial\beta)c + \frac{3}{2}\beta\partial c - 2b\gamma \ . \tag{10.1.21b}$$

Another free SCFT is the superconformal version of the linear dilaton theory. This has again the action (10.1.5), while

$$T_B(z) = -\frac{1}{\alpha'}\partial X^\mu \partial X_\mu + V_\mu \partial^2 X^\mu - \frac{1}{2}\psi^\mu \partial\psi_\mu \ , \tag{10.1.22a}$$

$$T_F(z) = i(2/\alpha')^{1/2}\psi^\mu \partial X_\mu - i(2\alpha')^{1/2} V_\mu \partial\psi^\mu \ , \tag{10.1.22b}$$

each having an extra term as in the bosonic case. The reader can verify that these satisfy the $N = 1$ algebra with

$$c = \frac{3}{2}D + 6\alpha' V^\mu V_\mu \ . \tag{10.1.23}$$

10.2 Ramond and Neveu–Schwarz sectors

We now study the spectrum of the $X^\mu\psi^\mu$ SCFT on a circle. Much of this is as in chapter 2, but the new ingredient is a more general periodicity condition. It is clearest to start with the cylindrical coordinate $w = \sigma^1 + i\sigma^2$. The matter fermion action

$$\frac{1}{4\pi}\int d^2w \left(\psi^\mu \partial_{\bar{w}}\psi_\mu + \tilde{\psi}^\mu \partial_w \tilde{\psi}_\mu\right) \tag{10.2.1}$$

must be invariant under the periodic identification of the cylinder, $w \cong w + 2\pi$. This condition plus Lorentz invariance still allows two possible periodicity conditions for ψ^μ,

$$\text{Ramond (R):} \quad \psi^\mu(w + 2\pi) = +\psi^\mu(w) \ , \tag{10.2.2a}$$

$$\text{Neveu–Schwarz (NS):} \quad \psi^\mu(w + 2\pi) = -\psi^\mu(w) \ , \tag{10.2.2b}$$

where the sign must be the same for all μ. Similarly there are two possible periodicities for $\tilde{\psi}^\mu$. Summarizing, we will write

$$\psi^\mu(w + 2\pi) = \exp(2\pi i\nu)\,\psi^\mu(w) \ , \tag{10.2.3a}$$

$$\tilde{\psi}^\mu(\bar{w} + 2\pi) = \exp(-2\pi i\tilde{\nu})\,\tilde{\psi}^\mu(\bar{w}) \ , \tag{10.2.3b}$$

where ν and $\tilde{\nu}$ take the values 0 and $\frac{1}{2}$.

Since we are initially interested in theories with the maximum Poincaré invariance, X^μ must be periodic. (Antiperiodicity of X^μ is interesting, and we have already encountered it for the twisted strings on an orbifold, but it would break some of the translation invariance.) The supercurrent then has the same periodicity as the corresponding ψ,

$$T_F(w + 2\pi) = \exp(2\pi i \nu)\, T_F(w)\ , \tag{10.2.4a}$$
$$\tilde{T}_F(\bar{w} + 2\pi) = \exp(-2\pi i \tilde{\nu})\, \tilde{T}_F(\bar{w})\ . \tag{10.2.4b}$$

Thus there are four different ways to put the theory on a circle, each of which will lead to a different Hilbert space — essentially there are four different kinds of closed superstring. We will denote these by $(\nu, \tilde{\nu})$ or by NS–NS, NS–R, R–NS, and R–R. They are analogous to the twisted and untwisted sectors of the $\mathbf{Z}_2$ orbifold. Later in the chapter we will see that consistency requires that the full string spectrum contain certain combinations of states from each sector.

To study the spectrum in a given sector expand in Fourier modes,

$$\psi^\mu(w) = i^{-1/2} \sum_{r \in \mathbf{Z}+\nu} \psi^\mu_r \exp(irw)\ , \quad \tilde{\psi}^\mu(\bar{w}) = i^{1/2} \sum_{r \in \mathbf{Z}+\tilde{\nu}} \tilde{\psi}^\mu_r \exp(-ir\bar{w})\ , \tag{10.2.5}$$

the phase factors being inserted to conform to convention later. On each side the sum runs over integers in the R sector and over (integers $+\ \frac{1}{2}$) in the NS sector. Let us also write these as Laurent expansions. Besides replacing $\exp(-iw) \to z$ we must transform the fields,

$$\psi^\mu_{z^{1/2}}(z) = (\partial_z w)^{1/2} \psi^\mu_{w^{1/2}}(w) = i^{1/2} z^{-1/2} \psi^\mu_{w^{1/2}}(w)\ . \tag{10.2.6}$$

The clumsy subscripts are a reminder that these transform with half the weight of a vector. Henceforth the frame will be indicated implicitly by the argument of the field. The Laurent expansions are then

$$\psi^\mu(z) = \sum_{r \in \mathbf{Z}+\nu} \frac{\psi^\mu_r}{z^{r+1/2}}\ , \quad \tilde{\psi}^\mu(\bar{z}) = \sum_{r \in \mathbf{Z}+\tilde{\nu}} \frac{\tilde{\psi}^\mu_r}{\bar{z}^{r+1/2}}\ . \tag{10.2.7}$$

Notice that in the NS sector, the branch cut in $z^{-1/2}$ offsets the original antiperiodicity, while in the R sector it introduces a branch cut. Let us also recall the corresponding bosonic expansions

$$\partial X^\mu(z) = -i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{m=-\infty}^{\infty} \frac{\alpha^\mu_m}{z^{m+1}}\ , \quad \bar{\partial} X^\mu(\bar{z}) = -i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{m=-\infty}^{\infty} \frac{\tilde{\alpha}^\mu_m}{\bar{z}^{m+1}}\ , \tag{10.2.8}$$

where $\alpha^\mu_0 = \tilde{\alpha}^\mu_0 = (\alpha'/2)^{1/2} p^\mu$ in the closed string and $\alpha^\mu_0 = (2\alpha')^{1/2} p^\mu$ in the open string.

The OPE and the Laurent expansions (or canonical quantization) give the anticommutators

$$\{\psi^\mu_r, \psi^\nu_s\} = \{\tilde{\psi}^\mu_r, \tilde{\psi}^\nu_s\} = \eta^{\mu\nu} \delta_{r,-s}\ , \tag{10.2.9a}$$
$$[\alpha^\mu_m, \alpha^\nu_n] = [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\eta^{\mu\nu} \delta_{m,-n}\ . \tag{10.2.9b}$$

For T_F and T_B the Laurent expansions are

$$T_F(z) = \sum_{r\in\mathbf{Z}+\nu} \frac{G_r}{z^{r+3/2}} \,, \quad \tilde{T}_F(\bar{z}) = \sum_{r\in\mathbf{Z}+\tilde{\nu}} \frac{\tilde{G}_r}{\bar{z}^{r+3/2}} \,, \tag{10.2.10a}$$

$$T_B(z) = \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}} \,, \quad \tilde{T}_B(\bar{z}) = \sum_{m=-\infty}^{\infty} \frac{\tilde{L}_m}{\bar{z}^{m+2}} \,. \tag{10.2.10b}$$

The usual CFT contour calculation gives the mode algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n} \,, \tag{10.2.11a}$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{c}{12}(4r^2 - 1)\delta_{r,-s} \,, \tag{10.2.11b}$$

$$[L_m, G_r] = \frac{m-2r}{2} G_{m+r} \,. \tag{10.2.11c}$$

This is known as the *Ramond algebra* for r, s integer and the *Neveu–Schwarz algebra* for r, s half-integer. The antiholomorphic fields give a second copy of these algebras.

The superconformal generators in either sector are

$$L_m = \frac{1}{2}\sum_{n\in\mathbf{Z}} {}^{\circ}_{\circ}\alpha^{\mu}_{m-n}\alpha_{\mu n}{}^{\circ}_{\circ} + \frac{1}{4}\sum_{r\in\mathbf{Z}+\nu}(2r-m)\,{}^{\circ}_{\circ}\psi^{\mu}_{m-r}\psi_{\mu r}{}^{\circ}_{\circ} + a^{\mathrm{m}}\delta_{m,0} \,, \tag{10.2.12a}$$

$$G_r = \sum_{n\in\mathbf{Z}} \alpha^{\mu}_{n}\psi_{\mu\, r-n} \,. \tag{10.2.12b}$$

Again ${}^{\circ}_{\circ}\ {}^{\circ}_{\circ}$ denotes creation–annihilation normal ordering. The normal ordering constant can be obtained by any of the methods from chapter 2; we will use here the mnemonic from the end of section 2.9. Each periodic boson contributes $-\frac{1}{24}$. Each periodic fermion contributes $+\frac{1}{24}$ and each antiperiodic fermion $-\frac{1}{48}$. Including the shift $+\frac{1}{24}c = \frac{1}{16}D$ gives

$$\mathrm{R:}\ \ a^{\mathrm{m}} = \frac{1}{16}D \,, \quad \mathrm{NS:}\ \ a^{\mathrm{m}} = 0 \,. \tag{10.2.13}$$

For the open string, the condition that the surface term in the equation of motion vanish allows the possibilities

$$\psi^{\mu}(0,\sigma^2) = \exp(2\pi i\nu)\,\tilde{\psi}^{\mu}(0,\sigma^2) \,, \quad \psi^{\mu}(\pi,\sigma^2) = \exp(2\pi i\nu')\,\tilde{\psi}^{\mu}(\pi,\sigma^2) \,. \tag{10.2.14}$$

By the redefinition $\tilde{\psi}^{\mu} \to \exp(-2\pi i\nu')\tilde{\psi}^{\mu}$, we can set $\nu' = 0$. There are therefore two sectors, R and NS, as compared to the four of the closed string. To write the mode expansion it is convenient to combine ψ^{μ} and $\tilde{\psi}^{\mu}$ into a single field with the extended range $0 \le \sigma^1 \le 2\pi$. Define

$$\psi^{\mu}(\sigma^1,\sigma^2) = \tilde{\psi}^{\mu}(2\pi - \sigma^1,\sigma^2) \tag{10.2.15}$$

for $\pi \le \sigma^1 \le 2\pi$. The boundary condition $\nu' = 0$ is automatic, and the antiholomorphicity of $\tilde{\psi}^{\mu}$ implies the holomorphicity of the extended ψ^{μ}.

Finally, the boundary condition (10.2.14) at $\sigma^1 = 0$ becomes a periodicity condition on the extended ψ^μ, giving one set of R or NS oscillators and the corresponding algebra.

NS and R spectra

We now consider the spectrum generated by a single set of NS or R modes, corresponding to the open string or to one side of the closed string. The NS spectrum is simple. There is no $r = 0$ mode, so we define the ground state to be annihilated by all $r > 0$ modes,

$$\psi^\mu_r|0\rangle_{\rm NS} = 0 \ , \quad r > 0 \ . \tag{10.2.16}$$

The modes with $r < 0$ then act as raising operators; since these are anticommuting, each mode can only be excited once.

The main point of interest is the R ground state, which is degenerate due to the ψ^μ_0s. Define the ground states to be those that are annihilated by all $r > 0$ modes. The ψ^μ_0 satisfy the Dirac gamma matrix algebra (10.1.4) with

$$\Gamma^\mu \cong 2^{1/2}\psi^\mu_0 \ . \tag{10.2.17}$$

Since $\{\psi^\mu_r, \psi^\nu_0\} = 0$ for $r > 0$, the ψ^μ_0 take ground states into ground states. The ground states thus form a representation of the gamma matrix algebra. This representation is worked out in section B.1; in $D = 10$ it has dimension 32. The reader who is not familiar with properties of spinors in various dimensions should read section B.1 at this point. We can take a basis of eigenstates of the Lorentz generators S_a, eq. (B.1.10):

$$|s_0, s_1, \ldots, s_4\rangle_{\rm R} \ \equiv \ |\mathbf{s}\rangle_{\rm R} \ , \quad s_a = \pm\tfrac{1}{2} \ . \tag{10.2.18}$$

The half-integral values show that these are indeed spacetime spinors. A more general basis for the spinors would be denoted $|\alpha\rangle_{\rm R}$. In the R sector of the open string not only the ground state but all states have half-integer spacetime spins, because the raising operators are vectors and change the S_a by integers. In the NS sector, the ground state is annihilated by $S^{\mu\nu}$ and is a Lorentz singlet, and all other states then have integer spin.

The Dirac representation **32** is reducible to two Weyl representations $\mathbf{16} + \mathbf{16}'$, distinguished by their eigenvalue under Γ as in eq. (B.1.11). This has a natural extension to the full string spectrum. The distinguishing property of Γ is that it anticommutes with all Γ^μ. Since the Dirac matrices are now the center-of-mass modes of ψ^μ, we need an operator that anticommutes with the full ψ^μ. We will call this operator

$$\exp(\pi i F) \ , \tag{10.2.19}$$

where F, the *world-sheet fermion number,* is defined only mod 2. Since ψ^μ changes F by one it anticommutes with the exponential. It is convenient to write F in terms of spacetime Lorentz generators, which in either sector of the ψ CFT are

$$\Sigma^{\mu\lambda} = -\frac{i}{2} \sum_{r \in \mathbf{Z}+\nu} [\psi_r^\mu, \psi_{-r}^\lambda] \ . \tag{10.2.20}$$

This is the natural extension of the zero-mode part (B.1.8). Define now

$$S_a = i^{\delta_{a,0}} \Sigma^{2a,2a+1} \ , \tag{10.2.21}$$

the i being included to make S_0 Hermitean, and let

$$F = \sum_{a=0}^{4} S_a \ . \tag{10.2.22}$$

This has the desired property. For example,

$$S_1(\psi_r^2 \pm i\psi_r^3) = (\psi_r^2 \pm i\psi_r^3)(S_1 \pm 1) \ , \tag{10.2.23}$$

so these oscillators change F by ± 1. The definition (10.2.22) makes it obvious that F is conserved by the OPE of the vertex operators, as a consequence of Lorentz invariance.[1] When we include the ghost part of the vertex operator in section 10.4, we will see that it contributes to the total F, so that on the total matter plus ghost ground state one has

$$\exp(\pi i F)|0\rangle_{\text{NS}} = -|0\rangle_{\text{NS}} \ , \tag{10.2.24a}$$
$$\exp(\pi i F)|\mathbf{s}\rangle_{\text{R}} = |\mathbf{s}'\rangle_{\text{R}} \Gamma_{\mathbf{s}'\mathbf{s}} \ . \tag{10.2.24b}$$

The ghost ground state contributes a factor -1 in the NS sector and $-i$ in the R sector.

Closed string spectra

In the closed string, the NS–NS states have integer spin. Because the spins S_a are additive, the half-integers from the two sides of the R–R sector also combine to give integer spin. The NS–R and R–NS states, on the other hand, have half-integer spin.

Let us look in more detail at the R–R sector, where the ground states $|\mathbf{s}, \mathbf{s}'\rangle_{\text{R}}$ are degenerate on both the right and left. They transform as the product of two Dirac representations, which is worked out in section B.1:

$$\begin{aligned} \mathbf{32}_{\text{Dirac}} \times \mathbf{32}_{\text{Dirac}} &= [0] + [1] + [2] + \ldots + [10] \\ &= [0]^2 + [1]^2 + \ldots + [4]^2 + [5] \ , \end{aligned} \tag{10.2.25}$$

[1] Lorentz invariance of the OPE holds separately for the ψ and X CFTs (and the $\tilde{\psi}$ CFT in the closed string) because they are decoupled from one another. However, the world-sheet supercurrent is only invariant under the overall Lorentz transformation.

Table 10.1. *$SO(9,1)$ representations of massless R–R states.*

$(\exp(\pi i F), \exp(\pi i \tilde{F}))$			$SO(9,1)$ rep.
$(+1,+1)$:	$\mathbf{16}\times\mathbf{16}$	$=$	$[1]+[3]+[5]_+$
$(+1,-1)$:	$\mathbf{16}\times\mathbf{16}'$	$=$	$[0]+[2]+[4]$
$(-1,+1)$:	$\mathbf{16}'\times\mathbf{16}$	$=$	$[0]+[2]+[4]$
$(-1,-1)$:	$\mathbf{16}'\times\mathbf{16}'$	$=$	$[1]+[3]+[5]_-$

where $[n]$ denotes an antisymmetric rank n tensor. For the closed string there are separate world-sheet fermion numbers F and $\tilde{F}$, which on the ground states reduce to the *chirality* matrices Γ and $\tilde{\Gamma}$ acting on the two sides. The ground states thus decompose as in table 10.1.

10.3 Vertex operators and bosonization

Consider first the unit operator. Fields remain holomorphic at the origin, and in particular they are single-valued. From the Laurent expansion (10.2.7), the single-valuedness means that the unit operator must be in the NS sector; the conformal transformation that takes the incoming string to the point $z=0$ cancels the branch cut from the antiperiodicity. The holomorphicity of ψ at the origin implies, via the contour argument, that the state corresponding to the unit operator satisfies

$$\psi^\mu_r|1\rangle = 0\ , \quad r=\frac{1}{2},\frac{3}{2},\dots\ , \tag{10.3.1}$$

and therefore

$$|1\rangle = |0\rangle\ . \tag{10.3.2}$$

Since the $\psi\psi$ OPE is single-valued, all products of ψ and its derivatives must be in the NS sector. The contour argument gives the map

$$\psi^\mu_{-r} \to \frac{1}{(r-1/2)!}\partial^{r-1/2}\psi^\mu(0)\ , \tag{10.3.3}$$

so that there is a one-to-one map between such products and NS states. The analog of the Noether relation (2.9.6) between the superconformal variation of an NS operator and the OPE is

$$\delta_\eta \mathscr{A}(z,\bar{z}) = -\epsilon\sum_{n=0}^{\infty}\frac{1}{n!}\Big[\partial^n\eta(z)G_{n-1/2} + (\partial^n\eta(z))^*\tilde{G}_{n-1/2}\Big]\cdot\mathscr{A}(z,\bar{z})\ . \tag{10.3.4}$$

The R sector vertex operators must be more complicated because the Laurent expansion (10.2.7) has a branch cut. We have encountered this before, for the winding state vertex operators in section 8.2 and the orbifold