Graded exercises in
Advanced level mathematics

Graded exercises in pure mathematics

Edited by Barrie Hunt

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0 Background knowledge

0.1 Basic arithmetic – highest common factor; lowest common multiple; fractions

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd} \qquad \frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$$

1 Express as a product of prime factors:

(a) 30 (b) 49 (c) 53 (d) 84
(e) 108 (f) 693 (g) 1144 (h) 14 553

2 Find the highest common factor (HCF) of:

(a) 6, 10 (b) 7, 14 (c) 30, 42
(d) 24, 40, 64 (e) 42, 70, 182 (f) 169, 234, 299
(g) 252, 378, 567 (h) 51, 527, 1343

3 Find the lowest common multiple (LCM) of:

(a) 6, 10 (b) 7, 14 (c) 30, 42 (d) 2, 3, 4
(e) 5, 25 (f) 5, 7, 11 (g) 4, 21, 22 (h) 14, 18, 21

4 Express each fraction in its lowest terms, without using a calculator:

(a) $\frac{7}{35}$ (b) $\frac{15}{125}$ (c) $\frac{26}{39}$ (d) $\frac{16}{80}$

(e) $\frac{81}{108}$ (f) $\dfrac{3a}{12a}$ (g) $\dfrac{42a^2}{56a}$ (h) $\dfrac{22ab^2}{121b}$

5 Complete:

(a) $\dfrac{3}{4}=\dfrac{}{24}$ (b) $\dfrac{4}{5}=\dfrac{}{20}$ (c) $\dfrac{4}{7}=\dfrac{}{21}$ (d) $\dfrac{7}{8}=\dfrac{}{64}$

(e) $\dfrac{7}{4}=\dfrac{}{20}$ (f) $\dfrac{2a}{3}=\dfrac{}{9}$ (g) $\dfrac{a}{b}=\dfrac{}{bx}$ (h) $\dfrac{2}{a}=\dfrac{}{a^2}$

6 Simplify, without using a calculator:

(a) $\dfrac{3}{4}+\dfrac{2}{3}$ (b) $\dfrac{2}{7}-\dfrac{1}{5}$ (c) $\dfrac{4}{13}+\dfrac{2}{7}$ (d) $\dfrac{5}{12}-\dfrac{3}{8}$

(e) $1\frac{3}{4}+2\frac{7}{8}$ (f) $5\frac{2}{3}-3\frac{1}{9}$ (g) $2\frac{1}{7}+\frac{3}{4}$ (h) $3\frac{2}{5}+2\frac{2}{3}$

7 Express as a single fraction:

(a) $\frac{3a}{4}+\frac{2a}{3}$ (b) $\frac{2a}{7}-\frac{a}{5}$ (c) $\frac{3}{a}+\frac{2}{a}$ (d) $\frac{3}{a}+\frac{2}{b}$

(e) $\frac{1}{u}+\frac{1}{v}$ (f) $\frac{5}{a}-\frac{2}{a^2}$ (g) $p-\frac{2}{q}$ (h) $\frac{3}{ab}-\frac{5}{ac}$

8 Without using a calculator, simplify and express each fraction in its lowest terms:

(a) $6\times\frac{2}{3}$ (b) $\frac{1}{2}\times\frac{3}{4}$ (c) $\frac{3}{5}\times\frac{4}{7}$ (d) $\frac{3}{5}\times\frac{4}{9}$

(e) $\frac{3a}{7}\times\frac{2}{5a}$ (f) $\frac{4a^2}{11}\times\frac{3}{2ab}$ (g) $x\times\frac{1}{x}$ (h) $x^2\left(\frac{3}{x}+\frac{2}{x^2}\right)$

9 Without using a calculator, simplify and express each fraction in its lowest terms:

(a) $6\div\frac{2}{3}$ (b) $\frac{1}{2}\div\frac{3}{4}$ (c) $\frac{3}{5}\div\frac{6}{25}$ (d) $\frac{3}{5}\div\frac{4}{9}$

(e) $\frac{3a}{7}\div\frac{2a}{5}$ (f) $\frac{4}{11a^2}\div\frac{2}{3ab}$ (g) $x\div\frac{1}{x}$ (h) $\frac{1}{x^2}\div\frac{1}{x}$

10 Which is larger, $\frac{77}{78}$ or $\frac{78}{79}$?

11 (a) The fraction $\frac{20}{91}$ is written as $\frac{1}{7}+\frac{1}{a}$. Find a.

(b) Calculate: (i) $(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{4})$

(ii) $(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{4})\dots(1-\frac{1}{n})$

12 Find the greatest number which, when divided into 1407 and 2140, leaves remainders of 15 and 23 respectively.

0.2 Laws of indices

$$a^m\times a^n=a^{m+n} \qquad \frac{a^m}{a^n}=a^{m-n} \qquad (a^m)^n=a^{mn}$$

1 Simplify:

(a) $a^3\times a^4$ (b) $a^7\times a^6$ (c) $a\times a^3$

(d) $2a^3\times 3a^2$ (e) $5a^2\times a^7$ (f) $\frac{2}{3}a^3\times 6a^4$

2 Simplify:

(a) $\frac{x^9}{x^2}$ (b) $\frac{p^4}{p^3}$ (c) $\frac{x^{12}}{x}$ (d) $\frac{12a^7}{4a^2}$ (e) $\frac{12a^5}{8a^3}$ (f) $\frac{2a^2b}{6ab^2}$

3 Simplify:

(a) $(a^5)^3$ (b) $(2a)^4$ (c) $(5a^3)^2$ (d) $5(a^3)^2$
(e) $(-2a^2)^4$ (f) $(3a^2b^3)^3$

4 Simplify:

(a) $\sqrt{x^2}$ (b) $\sqrt{x^6}$ (c) $\sqrt{a^2b^2}$ (d) $\sqrt{4a^2}$
(e) $\sqrt[3]{-x^6}$ (f) $\sqrt{9a^{10}b^4}$

5 Expand:

(a) $(1+x^2)^2$ (b) $(3-a^3)^2$ (c) $\left(x^2 - \dfrac{1}{x^2}\right)^2$

6 Simplify:

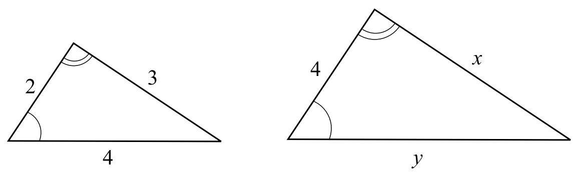
(a) $\dfrac{x^2+x^5}{x}$ (b) $\dfrac{3x^8+2x^4}{x^4}$

(c) $3x^2+(5x)^2-\dfrac{3x^3}{x}$ (d) $\dfrac{10x^2y+6xy^2-8x^2y^2}{2xy}$

0.3 Similar figures

1 Find the sides marked x and/or y in each of the following pairs of similar triangles.

(a)



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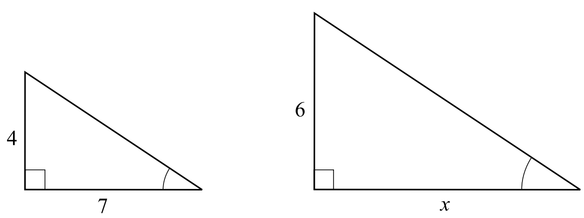

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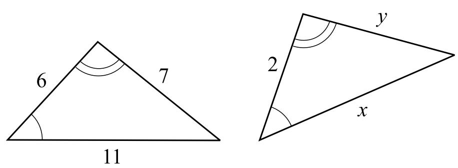

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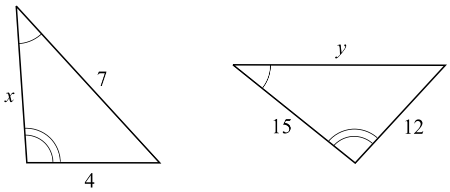

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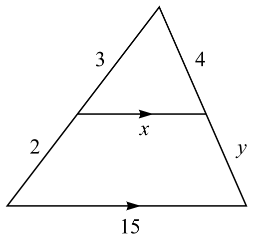

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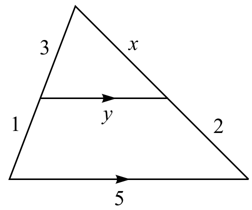
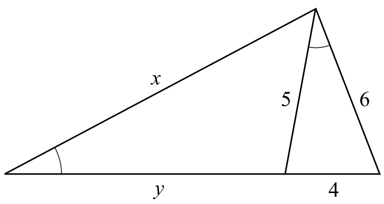
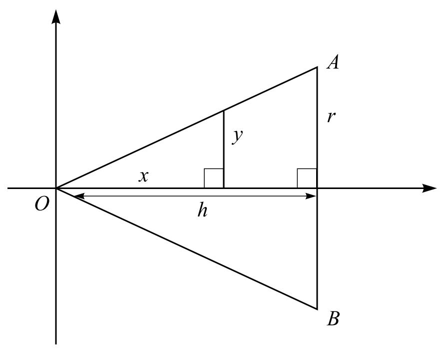


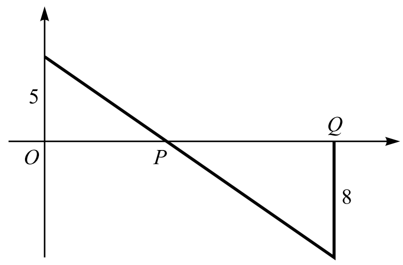
(g)



2 OAB is the cross-section of a cone, radius r, height h.
Express y in terms of r, h and x.



3 The coordinates of Q are (4, 0).
What are the coordinates of P?



4 A sphere has radius 8 cm and a second sphere has radius 12 cm.
What is the ratio of their (a) areas, (b) volumes?

5 A solid metal cylinder of radius 6 cm and height 12 cm weighs 6 kg. A second cylinder is made from the same material and has radius 8 cm and height 16 cm. How much does this cylinder weigh?

6 A liquid is poured into a hollow cone, which is placed with its vertex down. When 400 cm^3 has been poured in, the depth of water is 100 cm. What is the depth of water after (a) 1000 cm^3, (b) $x\,\text{cm}^3$ has been poured in? Plot the graph to show how depth varies with volume.

0.4 Basic algebra – multiplying brackets, factorising quadratics, solution of simultaneous equations

$$(a+b)(c+d) = ac + ad + bc + bd$$

1 Expand:

(a) $3(4+a)$ (b) $6(2-3a)$ (c) $a(a+3)$

(d) $a(2a+3b)$ (e) $3a(5a-2b)$ (f) $x\left(2+\frac{3}{x}\right)$

2 Multiply out the brackets:

(a) $(x+2)(x+5)$ (b) $(x-3)(x+4)$ (c) $(2x+1)(3x+5)$

(d) $(5x-2)(5x+2)$ (e) $(3a+2)^2$ (f) $(p+3q)(2p-5q)$

(g) $\left(x+\frac{2}{x}\right)^2$ (h) $(2x^2+1)(x+3)$

3 Factorise:

(a) $4x+8y$ (b) x^2-3x (c) $5x^2+2xy$

(d) $2\pi r^2+2\pi rh$ (e) $ut+\frac{1}{2}at^2$ (f) $2x^3+3x^4$

4 Factorise:

(a) x^2+4x+3 (b) x^2+2x-3 (c) a^2-6a+9

(d) $x^2+7x+10$ (e) p^2+p-30 (f) $2a^2+7a+3$

(g) $6y^2-7y-5$ (h) p^2-4q^2 (i) $p^2+4pq-12q^2$

(j) $15p^2-34pq-16q^2$ (k) $9x^2+30xy+25y^2$ (l) $10a^2+31a-14$

5 Simplify:

(a) $\dfrac{3x+6}{3}$ (b) $\dfrac{x^2+2x}{x}$ (c) $\dfrac{x^2+3x+2}{x+1}$ (d) $\dfrac{16-x^2}{x+4}$

6 Solve the simultaneous equations:

(a) $x+y=4$, $x-y=-6$ (b) $x+2y=8$, $x+5y=17$ (c) $2x+3y=2$, $x-2y=8$

(d) $3x - 2y = 1$, $-5x + 4y = 3$ (e) $2x + 5y = -14$, $3x + 2y = 1$ (f) $5x - 3y = 23$, $7x + 4y = -17$
(g) $4x - 3y = 0$, $6x + 15y = 13$ (h) $2x + 3y + 4 = 0$, $5x - y - 7 = 0$

7 Multiply out the brackets:

(a) $(x-1)(x^2+x+1)$ (b) $(a+b)^3$ (c) $(a+b)^4$
(d) $(x+\sqrt{2})(x-\sqrt{2})$

8 Simplify:

(a) $(a+b)^2 - (a-b)^2$ (b) $\dfrac{x^3+2x^2+x}{x^2+x}$ (c) $\dfrac{x^4-13x^2+36}{(x-2)(x^2-9)}$

9 Solve the pairs of simultaneous equations below, explaining your results graphically.

(a) $2x + 3y = 8$, $6x + 9y = 12$ (b) $2x + 3y = 8$, $6x + 9y = 24$

0.5 Solving equations; changing the subject of a formula

1 Solve the following equations.

(a) $2x + 1 = 7$ (b) $2 - 3x = 8$
(c) $5x + 2 = 3x - 5$ (d) $6x + 3 = 8 - 2x$
(e) $3(x+2) = 9x$ (f) $4(2x-7) = 3(5x+1)$
(g) $x^2 = 81$ (h) $x^2 - 25 = 0$
(i) $x = \dfrac{16}{x}$ (j) $x^3 + 27 = 0$
(k) $x^2 = 7x$ (l) $x - \dfrac{4}{x} = 0$
(m) $x(x-4) = 0$ (n) $(x+3)(x-7) = 0$
(o) $(2x-3)(x+4)(3x+2) = 0$

2 Rearrange to make the given variable the subject of the formula:

(a) $Q = CV$ (C) (b) $C = 2\pi r$ (r)
(c) $F = \frac{9}{5}C + 32$ (C) (d) $y = mx + c$ (m)
(e) $P = 2(\ell + w)$ (ℓ) (f) $S = \frac{1}{2}n(a + \ell)$ (a)
(g) $v^2 = u^2 + 2as$ (a) (h) $s = ut + \frac{1}{2}at^2$ (a)
(i) $u = a + (n-1)d$ (d) (j) $s = \frac{n}{2}\{2a + (n-1)d\}$ (d)

3 Rearrange to make the given variable the subject of the formula:

(a) $E = mc^2$ (c) (b) $V = \frac{4}{3}\pi r^3$ (r)

(c) $V = \frac{1}{3}\pi r^2 h$ (r) (d) $y = \dfrac{4}{x^2}$ (x)

(e) $I = \frac{1}{2}m(v^2 - u^2)$ (v) (f) $y = 2\sqrt{x} + 3$ (x)

(g) $T = 2\pi\sqrt{\dfrac{\ell}{g}}$ (ℓ) (h) $A = \pi(r^2 - r_1^2)$ (r)

(i) $y = \dfrac{1}{x - a}$ (x) (j) $c = \sqrt{a^2 + b^2}$ (a)

4 In each case, show clearly how the second formula may be obtained from the first.

(a) $I = \dfrac{iR}{R + r}$, $r = \dfrac{(i - I)R}{I}$

(b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $y = \dfrac{b}{a}\sqrt{(a^2 - x^2)}$

(c) $y = \dfrac{x - 2}{x}$, $x = \dfrac{2}{1 - y}$

(d) $y = \dfrac{3x + 2}{5 - x}$, $x = \dfrac{5y - 2}{y + 3}$

(e) $I = \dfrac{Er}{R + r}$, $r = \dfrac{IR}{E - I}$

(f) $\dfrac{1}{R} = \dfrac{1}{u} + \dfrac{1}{v}$, $v = \dfrac{Ru}{u - R}$

5 The surface area, S, of a cylinder is given by $S = 2\pi r^2 + 2\pi rh$.
Its volume, V, is given by $V = \pi r^2 h$. Express V in terms of S and r only.

0.6 The straight line $y = mx + c$; gradient and intercept

> The line $y = mx + c$ has gradient m, intercept c

1 Plot the graph of $y = 4x + 2$ for $-3 \le x \le 3$. Calculate the gradient of the line.
Write down where it crosses the y-axis (the y-intercept).

2 Complete the table.

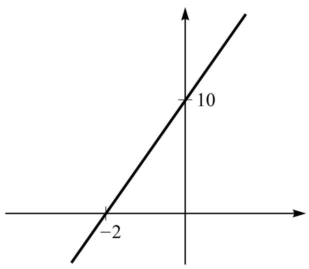
	Equation	Gradient	Intercept
(a)	$y = 5x - 2$		
(b)	$y = 1 - 3x$		
(c)	$y = \frac{1}{2}x$		
(d)	$y = -4 - 3x$		
(e)		2	5
(f)		6	−2
(g)		7	$\frac{1}{2}$
(h)		1	0
(i)	$2y = 4x + 1$		
(j)	$5y = 2x$		

3 Sketch the following lines.

(a) $y = 2x + 5$ (b) $y = \frac{1}{2}x + 2$ (c) $y = -x$ (d) $y = -x + 1$

4 Write down the equation of each of the lines shown.

(a)



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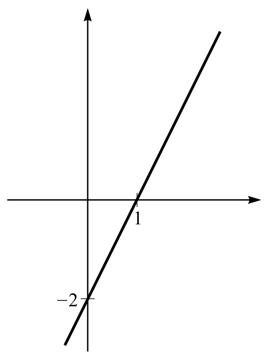

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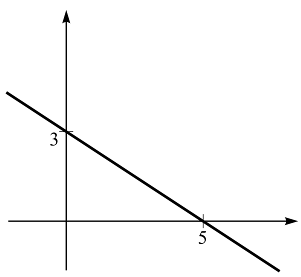

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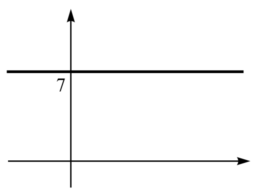

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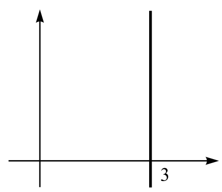
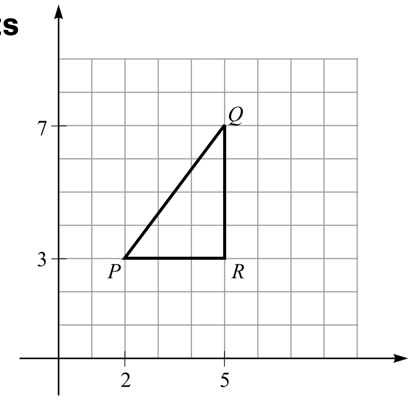


5 Find the equation of the line perpendicular to $y = 2x - 1$ which passes through (0, 3).

6 State the coordinates of the point where the line $\dfrac{y}{4} + \dfrac{x}{6} = 1$ crosses (a) the x-axis, (b) the y-axis.

0.7 The distance between two points



1 (a) P and Q are two points with coordinates (2, 3) and (5, 7) respectively. By applying Pythagoras' theorem to triangle PQR, find the distance PQ.
(b) By drawing a suitable diagram, find a formula for the distance PQ where P and Q have coordinates (x_1, y_1), (x_2, y_2) respectively.

2 Find the distance between the following pairs of points.

(a) (1, 2), (6, 14) (b) (3, 2), (6, 3) (c) (−1, 4), (2, 7)
(d) (4, 2), (1, −3)

3 Show that the triangle with vertices at (1, 0), (3, 0), $(2, \sqrt{3})$ is equilateral.

4 Which of the points (6, 4), (−3, 6), (2, −4) is nearest to (1, 2)?

5 Find the distance of the point $P(x, y)$ from (i) $O(0, 0)$ (ii) $R(4, 3)$.
If P is equidistant from O and R, find the equation of the locus of P.

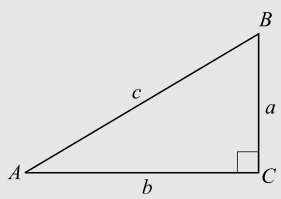
0.8 Trigonometry – right-angled triangles; sine and cosine rules

In right-angled triangles:

Pythagoras' theorem $a^2 + b^2 = c^2$

$\sin A = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{a}{c}, \quad \cos A = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{b}{c},$

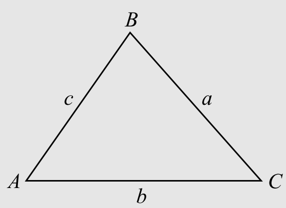
$\tan A = \dfrac{\text{opp}}{\text{adj}} = \dfrac{a}{b}$



In *all* triangles:

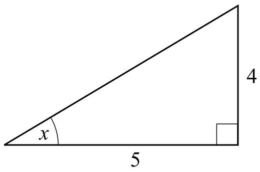
sine rule $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

cosine rule $a^2 = b^2 + c^2 - 2bc \cos A$



1 Find the angles marked x.

(a)



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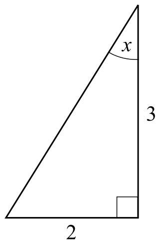

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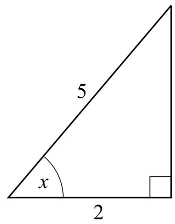

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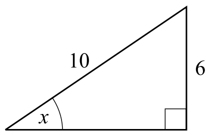
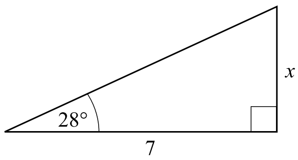


2 Find the sides marked x.

(a)



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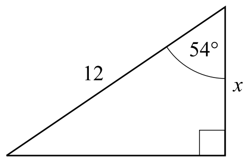

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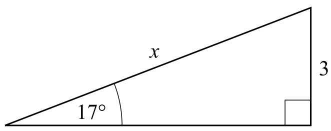

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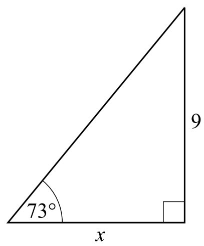

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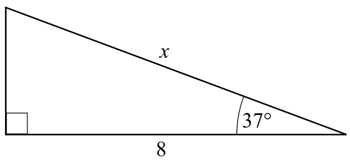
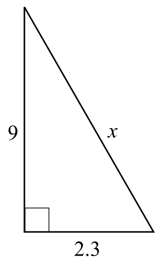
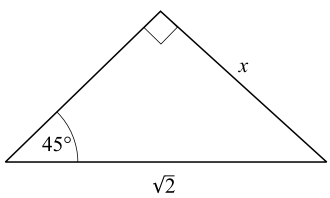


(f)



(g)



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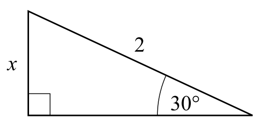
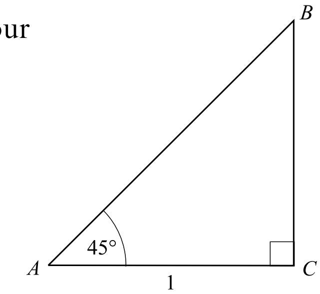


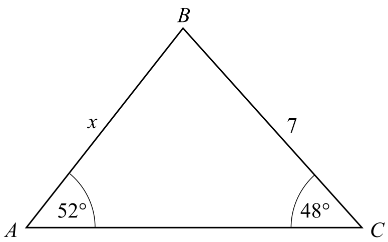
3 (a) Find the lengths of (i) BC, (ii) AB giving your answer in the form $\sqrt{a}$.

(b) Write down exact values for
(i) $\sin 45°$, (ii) $\cos 45°$, (iii) $\tan 45°$.

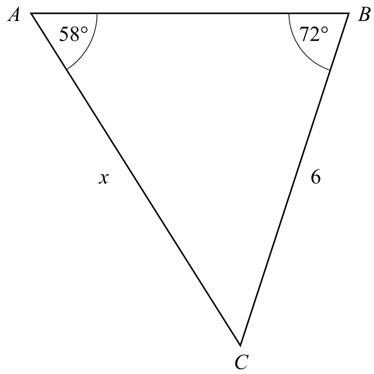


4 Use the sine rule to find the value of x.

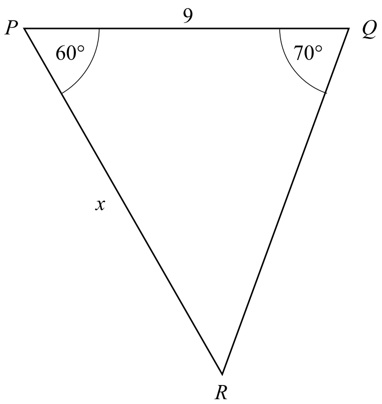
(a)



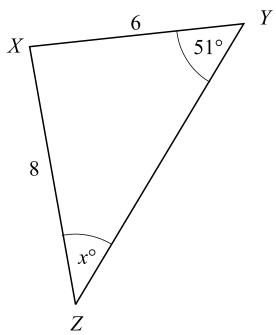
(b)



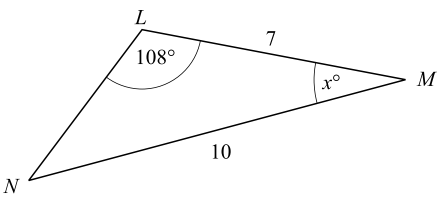
(c)



(d)



(e)



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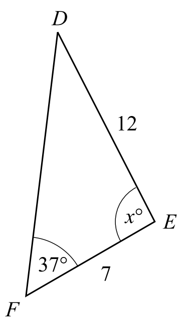
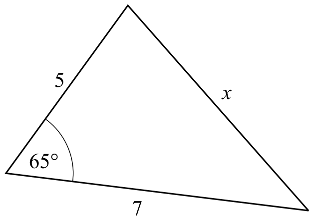


5 Use the cosine rule to find the value of x.

(a)



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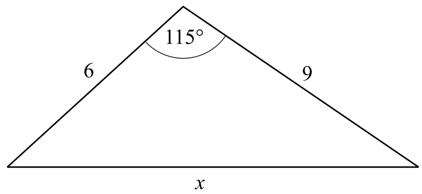

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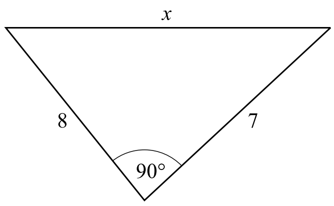

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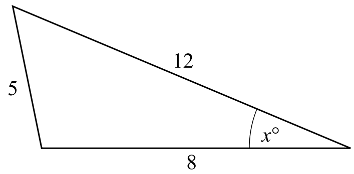

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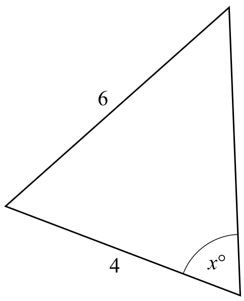

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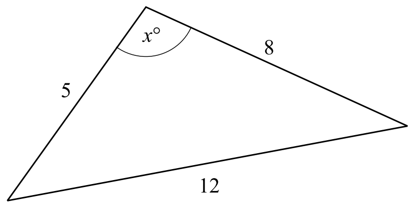
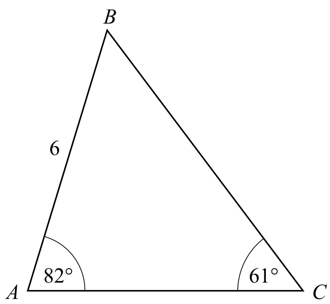
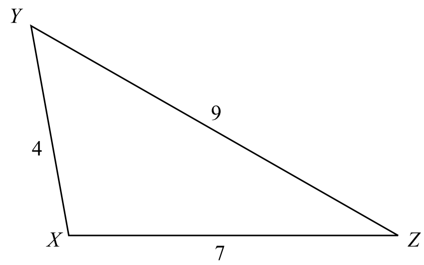


6 Use appropriate methods to find all sides and angles for:

(a)



(b)

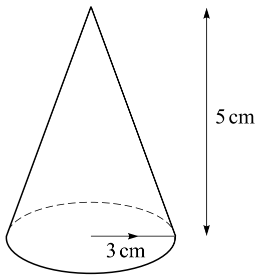


0.9 The cone and sphere

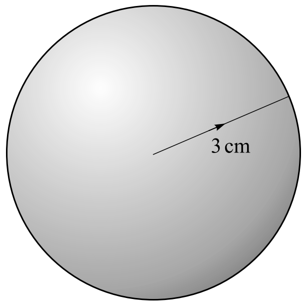
Volume of cone $= \frac{1}{3}\pi r^2 h$,	Volume of sphere $= \frac{4}{3}\pi r^3$
Surface area of cone $= \pi r\ell$,	Surface area of sphere $= 4\pi r^2$

1 Find the volumes of the following solid objects, giving your answers as multiples of π.

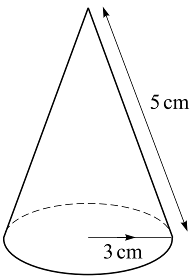
(a)



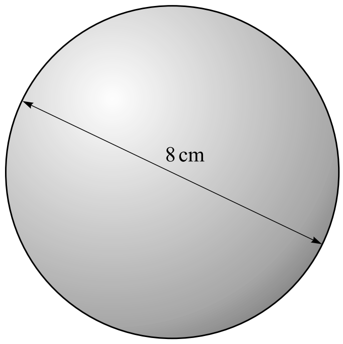
(b)



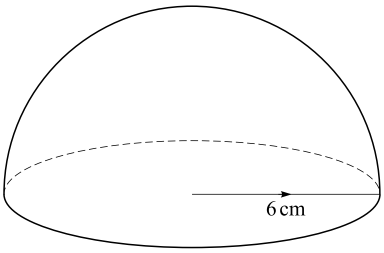
(c)



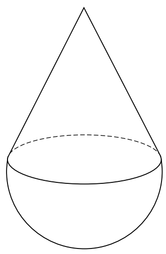
(d)



(e)



2 A child's toy is formed by attaching a cone to a hemisphere as shown. The radius of the hemisphere is 6 cm and the height of the toy is 14 cm. Find (a) its volume, (b) its surface area.



3 The earth may be treated as a sphere of radius 6370 km. Find (a) its surface area, (b) its volume.

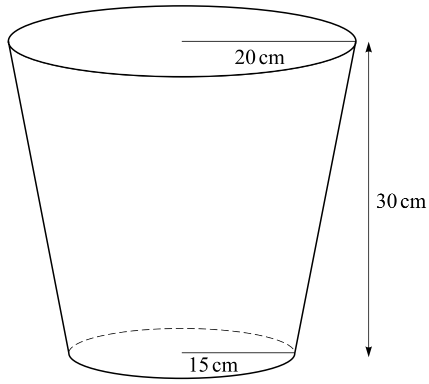
4 Twelve balls, each of radius 3 cm, are immersed in a cylinder of water, radius 10 cm, so that they are each fully submerged. What is the rise in the water level?

5 A solid metal cube of side 4 cm is melted down and recast as a sphere. Show that its radius is $\sqrt[3]{48/\pi}$.

6 A gas balloon, in the shape of a sphere, is made from 1000 m^2 of material. Estimate the volume of gas in the balloon.
What assumptions have you made?

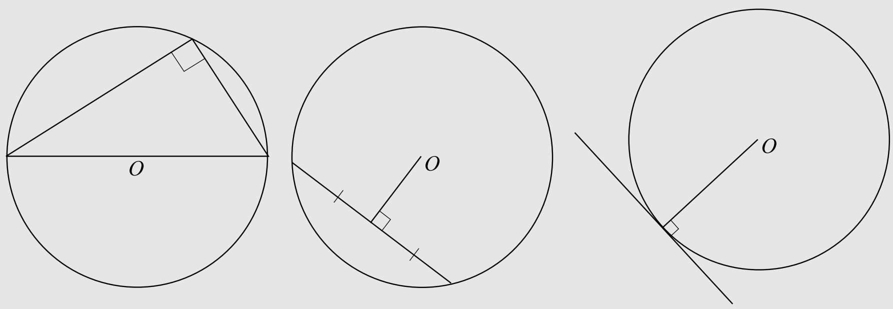
7 A hollow sphere has internal diameter 10 cm and external diameter 12 cm. What is the volume of the material used to make the sphere?

8 A bucket is in the shape of the frustrum of a cone. The radius of the base is 15 cm and the radius of the top is 20 cm. Find the volume of the bucket, given that its height is 30 cm.



0.10 Properties of a circle

Angle facts:



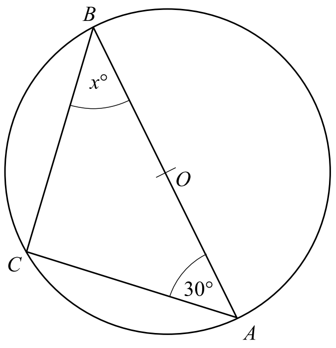
The angle in a semi-circle is a right angle.

The perpendicular from the centre to a chord bisects the chord.

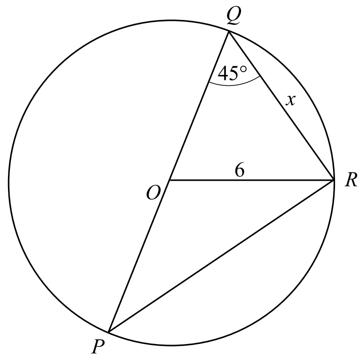
The radius is perpendicular to the tangent.

1 Find the value of x in each of the following.

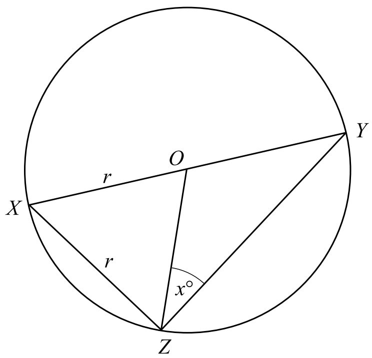
(a)



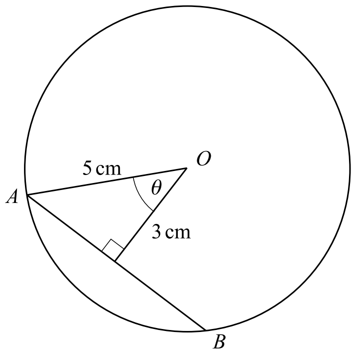
(b)



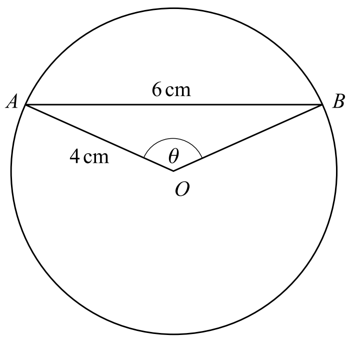
(c)



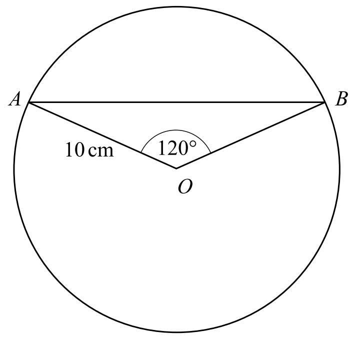
2 (a) AB is a chord of a circle, radius 5 cm, at a distance of 3 cm from the centre O. Find (i) the length AB, (ii) the angle θ.



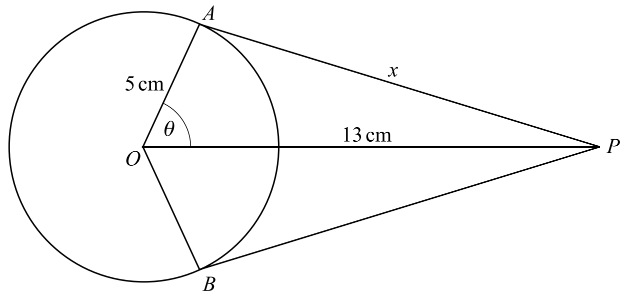
(b) Find the angle θ subtended by the chord AB in the diagram.



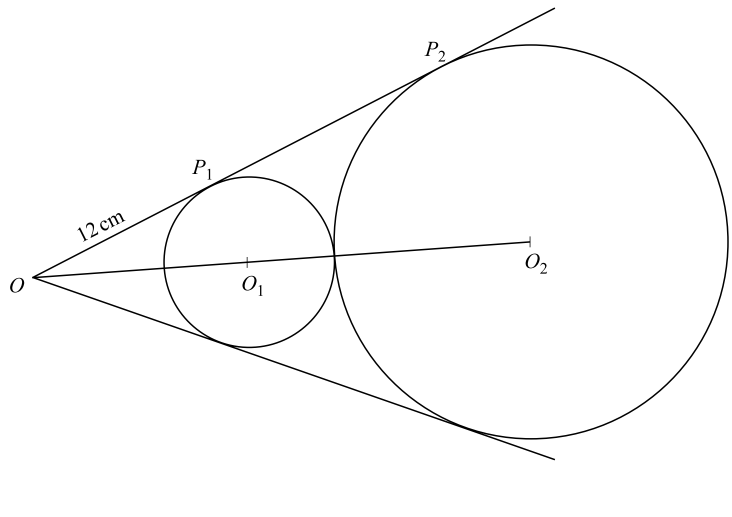
(c) Find the area of triangle AOB and hence find the area of the minor segment cut off by AB.



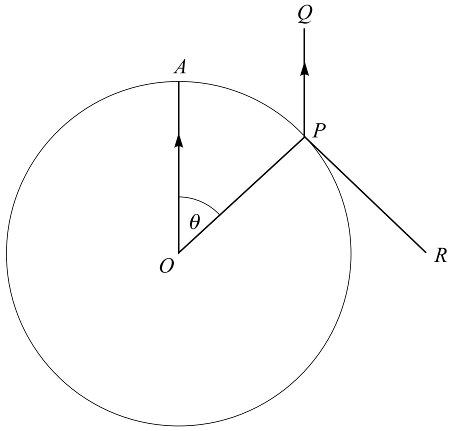
3 (a) AP and BP are tangents to the circle with centre O and radius 5 cm. $OP = 13$ cm. Find (i) AP, (ii) θ.



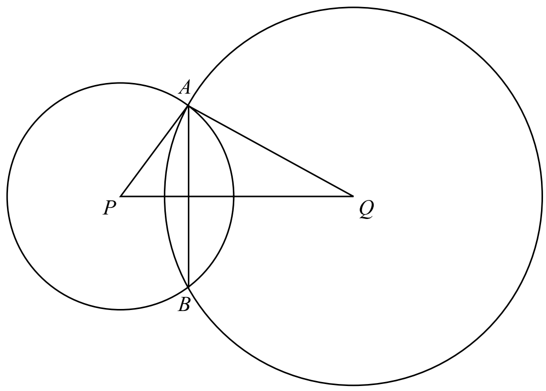
(b) OP_1P_2 is a tangent to two circles with centres O_1, O_2. $OP_1 = 12$ cm. The radius of the circle with centre O_1 is 5 cm. Find the radius of the circle with centre O_2.



(c) In the diagram, OA is parallel to PQ. Find the angle QPR in terms of θ.



4 Two circles, radii 3 cm and 5 cm, have centres P, Q respectively, $PQ = 7$ cm. If the circles intersect at A and B, find the length AB.



5 The distance from the Earth to the sun is 1.50×10^8 km. The diameter of the sun is 1.39×10^6 km.
Find the angle subtended by the sun from a point on the Earth.
What assumptions have you made?