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Advanced Level Mathematics

# Mechanics 2

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# Contents

# 1 The motion of projectiles

In this chapter the model of free motion under gravity is extended to objects projected at an angle. When you have completed it, you should

- understand displacement, velocity and acceleration as vector quantities
- be able to interpret the motion as a combination of the effects of the initial velocity and of gravity
- know that this implies the independence of horizontal and vertical motion
- be able to use equations of horizontal and vertical motion in calculations about the trajectory of a projectile
- know and be able to obtain general formulae for the greatest height, time of flight, range on horizontal ground and the equation of the trajectory
- be able to use your knowledge of trigonometry in solving problems.

Any object moving through the air will experience air resistance, and this is usually significant for objects moving at high speeds through large distances. The answers obtained in this chapter, which assume that air resistance is small and can be neglected, are therefore only approximate.

## 1.1 Velocity as a vector

When an object is thrown vertically upwards with initial velocity $u$, its displacement $s$ after time $t$ is given by the equation

$$s = ut - \tfrac{1}{2}gt^2,$$

where $g$ is the acceleration due to gravity.

One way to interpret this equation is to look at the two terms on the right separately. The first term, $ut$, would be the displacement if the object moved with constant velocity $u$, that is if there were no gravity. To this is added a term $\tfrac{1}{2}(-g)t^2$, which would be the displacement of the object in time $t$ if it were released from rest under gravity.

You can look at the equation

$$v = u - gt$$

in a similar way. Without gravity, the velocity would continue to have the constant value $u$ indefinitely. To this is added a term $(-g)t$, which is the velocity that the object would acquire in time $t$ if it were released from rest.

Now suppose that the object is thrown at an angle, so that it follows a curved path through the air. To describe this you can use the vector notation which you have already used (in M1 Chapter 10) for force. The symbol $\mathbf{u}$ written in bold stands for the velocity with which the object is thrown, that is a speed of magnitude $u$ in a given direction. If there were no gravity, then in time $t$ the object would have a displacement of magnitude $ut$ in that direction. It is natural to denote this by $\mathbf{u}t$, which is a vector displacement.

To this is added a vertical displacement of magnitude $\frac{1}{2}gt^2$ vertically downwards. In vector notation this can be written as $\frac{1}{2}\mathbf{g}t^2$, where the symbol $\mathbf{g}$ stands for an acceleration of magnitude $g$ in a direction vertically downwards.

To make an equation for this, let $\mathbf{r}$ denote the displacement of the object from its initial position at time $t = 0$. Then, assuming that air resistance can be neglected,

$$\mathbf{r} = \mathbf{u}t + \tfrac{1}{2}\mathbf{g}t^2 .$$

In this equation the symbol $+$ stands for vector addition, which is carried out by the triangle rule, the same rule that you use to add forces.

**Example 1.1.1**
A ball is thrown in the air with speed $12 \text{ m s}^{-1}$ at an angle of $70°$ to the horizontal. Draw a diagram to show where it is $1.5$ seconds later.

If there were no gravity, in 1.5 seconds the ball would have a displacement of magnitude $12 \times 1.5$ m, that is $18$ m, at $70°$ to the horizontal. This is represented by the arrow $\overrightarrow{OA}$ in Fig. 1.1, on a scale of 1 cm to 5 m. To this must be added a displacement of magnitude $\frac{1}{2} \times 10 \times 1.5^2$ m, that is $11.25$ m, vertically downwards, represented by the arrow $\overrightarrow{AB}$. The sum of these is the displacement $\overrightarrow{OB}$. So after $1.5$ seconds the ball is at $B$. You could if you wish calculate the coordinates of $B$, or the distance $OB$, but in this example these are not asked for.

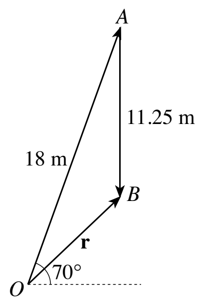


Fig. 1.1

**Example 1.1.2**
A stone is thrown from the edge of a cliff with speed $18 \text{ m s}^{-1}$. Draw diagrams to show the path of the stone in the next $4$ seconds if it is thrown
(a) horizontally, (b) at $30°$ to the horizontal.

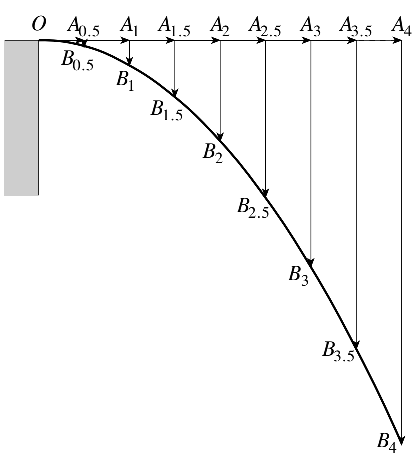


Fig. 1.2

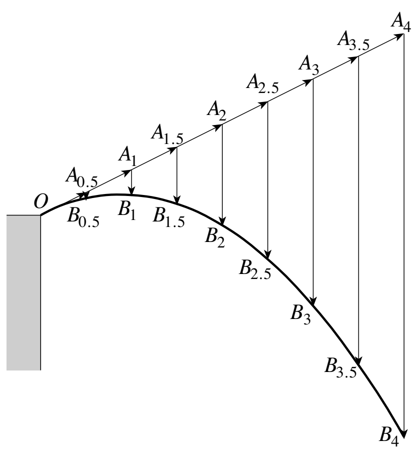


Fig. 1.3

These diagrams were produced by superimposing several diagrams like Fig. 1.1. In Figs. 1.2 and 1.3 (for parts (a) and (b) respectively) this has been done at intervals of 0.5 s, that is for $t = 0.5, 1, 1.5, \ldots, 4$. The displacements $\mathbf{u}t$ in these times have magnitudes 9 m, 18 m, … , 72 m. The vertical displacements have magnitudes 1.25 m, 5 m, 11.25 m, … , 80 m. The points corresponding to $A$ and $B$ at time $t$ are denoted by $A_t$ and $B_t$.

You can now show the paths by drawing smooth curves through the points $O$, $B_{0.5}$, $B_1$, … , $B_4$ for the two initial velocities.

The word **projectile** is often used to describe an object thrown in this way. The path of a projectile is called its **trajectory**.

A vector triangle can also be used to find the velocity of a projectile at a given time. If there were no gravity the velocity would have the constant value $\mathbf{u}$ indefinitely. The effect of gravity is to add to this a velocity of magnitude $gt$ vertically downwards, which can be written as the vector $\mathbf{g}t$. This gives the equation

$$\mathbf{v} = \mathbf{u} + \mathbf{g}t,$$

assuming that air resistance can be neglected.

**Example 1.1.3**
For the ball in Example 1.1.1, find the velocity after 1.5 seconds.

The vector $\mathbf{u}$ has magnitude 12 m s$^{-1}$ at 70° to the horizontal. The vector $\mathbf{g}t$ has magnitude $10 \times 1.5$ m s$^{-1}$, that is 15 m s$^{-1}$, directed vertically downwards.

To draw a vector triangle you need to choose a scale in which velocities are represented by displacements. Fig. 1.4 is drawn on a scale of 1 cm to 5 m s$^{-1}$. You can verify by measurement that the magnitude of $\mathbf{v}$ is about 5.5 m s$^{-1}$, and it is directed at about 42° below the horizontal.

Fig. 1.5 combines the results of Examples 1.1.1 and 1.1.3, showing both the position of the ball after 1.5 seconds and the direction in which it is moving.

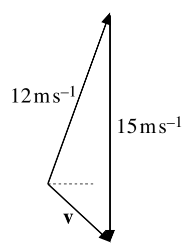


Fig. 1.4

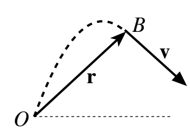


Fig. 1.5

## Exercise 1A

**1** A stone is thrown horizontally with speed 15 m s$^{-1}$ from the top of a cliff 30 metres high. Construct a diagram showing the positions of the particle at 0.5 second intervals. Estimate the distance of the stone from the thrower when it is level with the foot of the cliff, and the time that it takes to fall.

**2** A pipe discharges water from the roof of a building, at a height of 60 metres above the ground. Initially the water moves with speed $1 \text{ m s}^{-1}$, horizontally at right angles to the wall. Construct a diagram using intervals of 0.5 seconds to find the distance from the wall at which the water strikes the ground.

**3** A particle is projected with speed $10 \text{ m s}^{-1}$ at an angle of elevation of $40°$. Construct a diagram showing the position of the particle at intervals of 0.25 seconds for the first 1.5 seconds of its motion. Hence estimate the period of time for which the particle is higher than the point of projection.

**4** A ball is thrown with speed $14 \text{ m s}^{-1}$ at $35°$ above the horizontal. Draw diagrams to find the position and velocity of the ball 3 seconds later.

**5** A particle is projected with speed $9 \text{ m s}^{-1}$ at $40°$ to the horizontal. Calculate the time the particle takes to reach its maximum height, and find its speed at that instant.

**6** A cannon fires a shot at $38°$ above the horizontal. The initial speed of the cannonball is $70 \text{ m s}^{-1}$. Calculate the distance between the cannon and the point where the cannonball lands, given that the two positions are at the same horizontal level.

**7** A particle projected at $40°$ to the horizontal reaches its greatest height after 3 seconds. Calculate the speed of projection.

**8** A ball thrown with speed $18 \text{ m s}^{-1}$ is again at its initial height 2.7 seconds after projection. Calculate the angle between the horizontal and the initial direction of motion of the ball.

**9** A particle reaches its greatest height 2 seconds after projection, when it is travelling with speed $7 \text{ m s}^{-1}$. Calculate the initial velocity of the particle. When it is again at the same level as the point of projection, how far has it travelled horizontally?

**10** Two particles $A$ and $B$ are simultaneously projected from the same point on a horizontal plane. The initial velocity of $A$ is $15 \text{ m s}^{-1}$ at $25°$ to the horizontal, and the initial velocity of $B$ is $15 \text{ m s}^{-1}$ at $65°$ to the horizontal.

(a) Construct a diagram showing the paths of both particles until they strike the horizontal plane.

(b) From your diagram estimate the time that each particle is in the air.

(c) Calculate these times, correct to 3 significant figures.

## 1.2 Coordinate methods

For the purposes of calculation it often helps to use coordinates, with column vectors representing displacements, velocities and accelerations, just as was done for forces in M1 Chapter 10. It is usual to take the $x$-axis horizontal and the $y$-axis vertical.

For instance, in Example 1.1.2(a), the initial velocity $\mathbf{u}$ of the stone was $18 \text{ m s}^{-1}$ horizontally, which could be represented by the column vector $\begin{pmatrix} 18 \\ 0 \end{pmatrix}$. Since the units are metres and seconds, $\mathbf{g}$ is $10 \text{ m s}^{-2}$ vertically downwards, represented by $\begin{pmatrix} 0 \\ -10 \end{pmatrix}$.

Denoting the displacement $\mathbf{r}$ by $\begin{pmatrix} x \\ y \end{pmatrix}$, the equation $\mathbf{r} = \mathbf{u}t + \frac{1}{2}\mathbf{g}t^2$ becomes

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 18 \\ 0 \end{pmatrix} t + \tfrac{1}{2}\begin{pmatrix} 0 \\ -10 \end{pmatrix} t^2, \quad \text{or more simply} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 18t \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -5t^2 \end{pmatrix} = \begin{pmatrix} 18t \\ -5t^2 \end{pmatrix}.$$

You can then read off along each line to get the pair of equations

$$x = 18t \qquad \text{and} \qquad y = -5t^2.$$

From these you can calculate the coordinates of the stone after any time $t$.

You can turn the first equation round as $t = \frac{1}{18}x$ and then substitute this in the second equation to get $y = -5\left(\frac{1}{18}x\right)^2$, or (approximately) $y = -0.015x^2$. This is the equation of the trajectory. You will recognise this as a parabola with its vertex at $O$.

You can do the same thing with the velocity equation $\mathbf{v} = \mathbf{u} + \mathbf{g}t$, which becomes

$$\mathbf{v} = \begin{pmatrix} 18 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -10 \end{pmatrix} t = \begin{pmatrix} 18 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -10t \end{pmatrix} = \begin{pmatrix} 18 \\ -10t \end{pmatrix}.$$

This shows that the velocity has components $18$ and $-10t$ in the $x$- and $y$-directions respectively.

Notice that $18$ is the derivative of $18t$ with respect to $t$, and $-10t$ is the derivative of $-5t^2$. This is a special case of a general rule.

> If the displacement of a projectile is $\begin{pmatrix} x \\ y \end{pmatrix}$, its velocity is $\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}$.

This is a generalisation of the result given in M1 Section 11.2 for motion in a straight line.

Here is a good place to use the shorthand notation (dot notation) introduced in M1 Section 11.5, using $\dot{x}$ to stand for $\frac{dx}{dt}$ and $\dot{y}$ for $\frac{dy}{dt}$. You can then write the velocity vector as $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$.

Now consider the general case, when the projectile starts with an initial speed $u$ at an angle $\theta$ to the horizontal. Its initial velocity $\mathbf{u}$ can be described either in terms of $u$ and $\theta$, or in terms of its horizontal and vertical components $p$ and $q$. These are connected by $p = u\cos\theta$ and $q = u\sin\theta$. The notation is illustrated in Figs. 1.6 and 1.7.

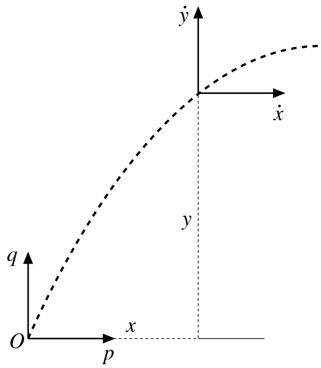



Fig. 1.6

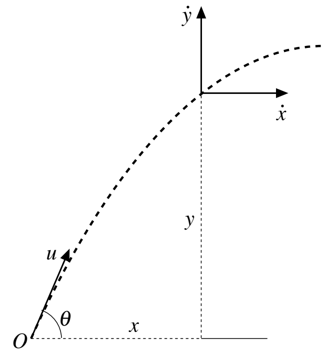



Fig. 1.7

The acceleration $\mathbf{g}$ is represented by $\begin{pmatrix} 0 \\ -g \end{pmatrix}$, so the equation

$$\mathbf{r} = \mathbf{u}t + \tfrac{1}{2}\mathbf{g}t^2$$

becomes

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} pt \\ qt \end{pmatrix} + \begin{pmatrix} 0 \\ -\tfrac{1}{2}gt^2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u\cos\theta\, t \\ u\sin\theta\, t \end{pmatrix} + \begin{pmatrix} 0 \\ -\tfrac{1}{2}gt^2 \end{pmatrix}.$$

By reading along each line in turn, the separate equations for the coordinates are

$$x = pt \quad \text{or} \quad x = u\cos\theta\, t,$$

and $\quad y = qt - \tfrac{1}{2}gt^2 \quad$ or $\quad y = u\sin\theta\, t - \tfrac{1}{2}gt^2.$

In a similar way, $\mathbf{v} = \mathbf{u} + \mathbf{g}t$ becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} 0 \\ -gt \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} u\cos\theta \\ u\sin\theta \end{pmatrix} + \begin{pmatrix} 0 \\ -gt \end{pmatrix}.$$

So

$$\dot{x} = p \quad \text{or} \quad \dot{x} = u\cos\theta,$$

and $\quad \dot{y} = q - gt \quad$ or $\quad \dot{y} = u\sin\theta - gt.$

Since $g$, $p$, $q$, $u$ and $\theta$ are all constant, you can see again that $\dot{x}$ and $\dot{y}$ are the derivatives of $x$ and $y$ with respect to $t$.

Now the equations $x = pt$ and $\dot{x} = p$ are just the same as those you would use for a particle moving in a straight line with constant velocity $p$. And the equations $y = qt - \tfrac{1}{2}gt^2$ and $\dot{y} = q - gt$ are the same as those for a particle moving in a vertical line with initial velocity $q$ and acceleration $-g$. This establishes the **independence of horizontal and vertical motion**.

> If a projectile is launched from $O$ with an initial velocity having horizontal and vertical components $p$ and $q$, under the action of the force of gravity alone and neglecting air resistance, and if its coordinates at a later time are $(x, y)$, then
>
> - the value of $x$ is the same as for a particle moving in a horizontal line with constant velocity $p$;
> - the value of $y$ is the same as for a particle moving in a vertical line with initial velocity $q$ and acceleration $-g$.

**Example 1.2.1**

A golf ball is driven with a speed of $45 \text{ m s}^{-1}$ at $37°$ to the horizontal across horizontal ground. How high above the ground does it rise, and how far away from the starting point does it first land?

To a good enough approximation $\cos 37° = 0.8$ and $\sin 37° = 0.6$, so the horizontal and vertical components of the initial velocity are $p = 45 \times 0.8 \text{ m s}^{-1} = 36 \text{ m s}^{-1}$ and $q = 45 \times 0.6 \text{ m s}^{-1} = 27 \text{ m s}^{-1}$. The approximate value of $g$ is $10 \text{ m s}^{-2}$.

To find the height you only need to consider the $y$-coordinate. To adapt the equation $v^2 = u^2 + 2as$ with the notation of Fig. 1.6, you have to insert the numerical values $u$ (that is $q$) = 27 and $a = -10$, and replace $s$ by $y$ and $v$ by $\dot{y}$. This gives

$$\dot{y}^2 = 27^2 - 2 \times 10 \times y = 729 - 20y.$$

When the ball is at its greatest height, $\dot{y} = 0$, so $729 - 20y = 0$. This gives $y = \frac{729}{20} = 36.45$.

To find how far away the ball lands you need to use both coordinates, and the link between these is the time $t$. So use the $y$-equation to find how long the ball is in the air, and then use the $x$-equation to find how far it goes horizontally in that time.

Adapting the equation $s = ut + \frac{1}{2}at^2$ for the vertical motion,

$$y = 27t - 5t^2.$$

When the ball hits the ground $y = 0$, so that $t = \frac{27}{5} = 5.4$. A particle moving horizontally with constant speed $36 \text{ m s}^{-1}$ would go $36 \times 5.4$ m, that is $194.4$ m, in this time.

So, according to the gravity model, the ball would rise to a height of about 36 metres, and first land about 194 metres from the starting point.

In practice, these answers would need to be modified to take account of air resistance and the aerodynamic lift on the ball.

**Example 1.2.2**

In a game of tennis a player serves the ball horizontally from a height of $2$ metres. It has to satisfy two conditions.

(i) It must pass over the net, which is $0.9$ metres high at a distance of $12$ metres from the server.

(ii) It must hit the ground less than $18$ metres from the server.

At what speeds can it be hit?

It is simplest to take the origin at ground level, rather than at the point from which the ball is served, so add $2$ to the $y$-coordinate given by the general formula. If the initial speed of the ball is $p \text{ m s}^{-1}$,

$$x = pt \quad \text{and} \quad y = 2 - 5t^2.$$

Both conditions involve both the $x$- and $y$-coordinates, and the time $t$ is used as the link.

(i) The ball passes over the net when $12 = pt$, that is $t = \frac{12}{p}$. The value of $y$ is then $2 - 5\left(\frac{12}{p}\right)^2 = 2 - \frac{720}{p^2}$, and this must be more than $0.9$. So $2 - \frac{720}{p^2} > 0.9$. This gives $\frac{720}{p^2} < 1.1$, which is $p > \sqrt{\frac{720}{1.1}} \approx 25.6$.

(ii) The ball lands when $y = 0$, that is when $2 - 5t^2 = 0$, or $t = \sqrt{\frac{2}{5}}$. It has then gone a horizontal distance of $p\sqrt{\frac{2}{5}}$ metres, and to satisfy the second condition you need $p\sqrt{\frac{2}{5}} < 18$. This gives $p < 18\sqrt{\frac{5}{2}} \approx 28.5$.

So the ball can be hit with any speed between about $25.6 \text{ m s}^{-1}$ and $28.5 \text{ m s}^{-1}$.

**Example 1.2.3**

A cricketer scores a six by hitting the ball at an angle of $30°$ to the horizontal. The ball passes over the boundary $90$ metres away at a height of $5$ metres above the ground. Neglecting air resistance, find the speed with which the ball was hit.

If the initial speed was $u \text{ m s}^{-1}$, the equations of horizontal and vertical motion are

$$x = u\cos 30°\, t \quad \text{and} \quad y = u\sin 30° t - 5t^2.$$

You know that, when the ball passes over the boundary, $x = 90$ and $y = 5$. Using the values $\cos 30° = \frac{1}{2}\sqrt{3}$ and $\sin 30° = \frac{1}{2}$,

$$90 = u \times \tfrac{1}{2}\sqrt{3} \times t = \tfrac{1}{2}\sqrt{3}ut \quad \text{and} \quad 5 = u \times \tfrac{1}{2} \times t - 5t^2 = \tfrac{1}{2}ut - 5t^2$$

for the same value of $t$.

From the first equation, $ut = \frac{180}{\sqrt{3}} = 60\sqrt{3}$. Substituting this in the second equation gives $5 = 30\sqrt{3} - 5t^2$, which gives $t = \sqrt{6\sqrt{3} - 1} = 3.06\ldots$.

It follows that $u = \frac{60\sqrt{3}}{t} = \frac{60\sqrt{3}}{3.06\ldots} \approx 33.9$.

The initial speed of the ball was about $34 \text{ m s}^{-1}$.

**Example 1.2.4**

A boy uses a catapult to send a small ball through his friend's open window. The window is 8 metres up a wall 12 metres away from the boy. The ball enters the window descending at an angle of $45°$ to the horizontal. Find the initial velocity of the ball.

Denote the horizontal and vertical components of the initial velocity by $p \text{ m s}^{-1}$ and $q \text{ m s}^{-1}$. If the ball enters the window after $t$ seconds,

$$12 = pt \quad \text{and} \quad 8 = qt - 5t^2.$$

Also, as the ball enters the window, its velocity has components $\dot{x} = p$ and $\dot{y} = q - 10t$. Since this is at an angle of $45°$ below the horizontal, $\dot{y} = -\dot{x}$, so $q - 10t = -p$, or

$$p + q = 10t.$$

You now have three equations involving $p$, $q$ and $t$. From the first two equations, $p = \frac{12}{t}$ and $q = \frac{8 + 5t^2}{t}$. Substituting these expressions in the third equation gives $\frac{12}{t} + \frac{8 + 5t^2}{t} = 10t$, that is

$$12 + \left(8 + 5t^2\right) = 10t^2, \quad \text{which simplifies to} \quad 5t^2 = 20.$$

So $t = 2$, from which you get $p = \frac{12}{2} = 6$ and $q = \frac{8 + 5 \times 2^2}{2} = 14$.

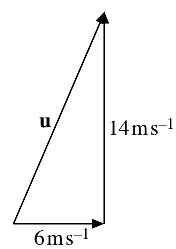


Fig. 1.8

Fig. 1.8 shows how these components are combined by the triangle rule to give the initial velocity of the ball. This has magnitude $\sqrt{6^2 + 14^2} \text{ m s}^{-1} \approx 15.2 \text{ m s}^{-1}$ at an angle $\tan^{-1} \frac{14}{6} \approx 66.8°$ to the horizontal.

The ball is projected at just over $15 \text{ m s}^{-1}$ at $67°$ to the horizontal.

## Exercise 1B

Assume that all motion takes place above a horizontal plane.

**1** A particle is projected horizontally with speed $13 \text{ m s}^{-1}$, from a point high above a horizontal plane. Find the horizontal and vertical components of the velocity of the particle after 2 seconds.

**2** The time of flight of an arrow fired with initial speed $30 \text{ m s}^{-1}$ horizontally from the top of a tower was 2.4 seconds. Calculate the horizontal distance from the tower to the arrow's landing point. Calculate also the height of the tower.

**3** Show that the arrow in Question 2 enters the ground with a speed of about $38 \text{ m s}^{-1}$ at an angle of about 39° to the horizontal.

**4** A stone is thrown from the point $O$ on top of a cliff with velocity $\begin{pmatrix} 15 \\ 0 \end{pmatrix} \text{ m s}^{-1}$. Find the position vector of the stone after 2 seconds.

**5** A particle is projected with speed $35 \text{ m s}^{-1}$ at an angle of $40°$ above the horizontal. Calculate the horizontal and vertical components of the displacement of the particle after 3 seconds. Calculate also the horizontal and vertical components of the velocity of the particle at this instant.

**6** A famine relief aircraft, flying over horizontal ground at a height of 245 metres, drops a sack of food.

(a) Calculate the time that the sack takes to fall.

(b) Calculate the vertical component of the velocity with which the sack hits the ground.

(c) If the speed of the aircraft is $70 \text{ m s}^{-1}$, at what distance before the target zone should the sack be released?

**7** A girl stands at the water's edge and throws a flat stone horizontally from a height of $80 \text{ cm}$.

(a) Calculate the time the stone is in the air before it hits the water.

(b) Find the vertical component of the velocity with which the stone hits the water.

The girl hopes to get the stone to bounce off the water surface. To do this the stone must hit the water at an angle to the horizontal of $15°$ or less.

(c) What is the least speed with which she can throw the stone to achieve this?

(d) If she throws the stone at this speed, how far away will the stone hit the water?

**8** A batsman tries to hit a six, but the ball is caught by a fielder on the boundary. The ball is in the air for 3 seconds, and the fielder is 60 metres from the bat. Calculate

(a) the horizontal component, (b) the vertical component

of the velocity with which the ball is hit.

Hence find the magnitude and direction of this velocity.

**9** A stone thrown with speed $17 \text{ m s}^{-1}$ reaches a greatest height of 5 metres. Calculate the angle of projection.

**10** A particle projected at $30°$ to the horizontal rises to a height of 10 metres. Calculate the initial speed of the particle, and its least speed during the flight.

**11** In the first 2 seconds of motion a projectile rises 5 metres and travels a horizontal distance of 30 metres. Calculate its initial speed.

**12** The nozzle of a fountain projects a jet of water with speed $12 \text{ m s}^{-1}$ at $70°$ to the horizontal. The water is caught in a cup 5 metres above the level of the nozzle. Calculate the time taken by the water to reach the cup.

**13** A stone was thrown with speed $15 \text{ m s}^{-1}$, at an angle of $40°$. It broke a small window 1.2 seconds after being thrown. Calculate the distance of the window from the point at which the stone was thrown.

**14** A football, kicked from ground level, enters the goal after 2 seconds with velocity $\begin{pmatrix} 12 \\ -9 \end{pmatrix} \text{m s}^{-1}$. Neglecting air resistance, calculate

(a) the speed and angle at which the ball was kicked,

(b) the height of the ball as it enters the goal,

(c) the greatest height of the ball above the ground.

**15** A golfer strikes a ball in such a way that it leaves the point $O$ at an angle of elevation of $\theta°$ and with speed of $40 \text{ m s}^{-1}$. After 5 seconds the ball is at a point $P$ which is 120 metres horizontally from $O$ and $h$ metres above $O$. Assuming that the flight of the ball can be modelled by the motion of a particle with constant acceleration, find the value of $\theta$ and the value of $h$. Give one force on the golf ball that this model does not allow for.

(OCR)

**16** A tennis player hits the ball towards the net with velocity $\begin{pmatrix} 15 \\ 5 \end{pmatrix} \text{m s}^{-1}$ from a point 8 metres from the net and 0.4 metres above the ground. The ball is in the air for 0.8 seconds before hitting the opponent's racket. Find, at the instant of impact with the opponent's racket,

(a) the velocity of the ball,

(b) the distance of the ball from the net and its height above the ground.

**17** A projectile reaches its greatest height after 2 seconds, when it is 35 metres from its point of projection. Determine the initial velocity.

**18** A tennis ball $A$ is dropped from the top of a vertical tower which is 15 metres high. At time $t$ seconds later the tennis ball is at a height $h$ metres above the ground. Assuming that the motion may be modelled by that of a particle, express $h$ in terms of $t$.

The point $O$ is 20 metres from the tower and is at the same horizontal level as the foot of the tower. At the instant that $A$ is dropped, a second ball $B$ is projected from $O$, towards the tower, with speed $V \text{ m s}^{-1}$ at an angle of elevation $\theta°$. The motion of $B$, before any impact, may be modelled by that of a particle, and takes place in a vertical plane containing the tower. Given that the two balls collide when $t = 1.5$ and before $B$ hits the ground, show that $V \sin\theta = 10$ and find the value of $V \cos\theta$.

Deduce the values of $V$ and $\theta$.

(OCR)

**19** The behaviour of a water droplet in an ornamental fountain is modelled by the motion of a particle moving freely under gravity with constant acceleration. The particle is projected from the point $O$ with speed $9.1 \text{ m s}^{-1}$, in a direction which makes an angle of $76°$ with the horizontal. The horizontal and vertical displacements of the particle from $O$ after $t$ seconds are $x$ metres and $y$ metres respectively.

(a) Write down expressions for $x$ and $y$ in terms of $t$ and hence show that $y = 4x - x^2$ is an approximation for the equation of the path of the particle.

(b) Find the range of the particle on the horizontal plane through $O$.

(c) Make a sketch showing the path of a particle.

The point of projection $O$ is in the surface of a pond. The water droplets are required to hit the surface of the pond at points on a circle with centre $O$ and diameter 7 metres. This requirement can be met, without changing the speed of projection, either by increasing the angle of projection slightly (say to about $77\frac{1}{2}°$) or by reducing it considerably (say to about $12\frac{1}{2}°$). In a single diagram sketch the path of the droplet in each of these cases.

(OCR, adapted)

**20** A ski-jumper takes off from the ramp travelling at an angle of $10°$ below the horizontal with speed 72 kilometres per hour. Before landing she travels a horizontal distance of 70 metres. Find the time she is in the air, and the vertical distance she falls.

## 1.3 Some general formulae

Some of the results in this section use advanced trigonometry from P2&3 Chapter 5.

When you have more complicated problems to solve, it is useful to know formulae for some of the standard properties of trajectories. These are given in the notation of Fig. 1.7, which is repeated here as Fig. 1.9.

The formulae are based on the assumption that $O$ is at ground level. If not, adjustments will be needed to allow for this.

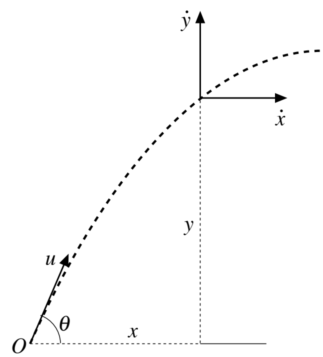



Fig. 1.9

**(i) Greatest height**

This depends only on the vertical motion of the projectile, for which the component of the initial velocity is $u\sin\theta$. The greatest height is reached when the vertical component of velocity is $0$. By the usual constant acceleration formula, if $h$ is the greatest height,

$$0^2 = (u\sin\theta)^2 - 2gh.$$

Therefore

$$h = \frac{u^2 \sin^2\theta}{2g}.$$

**(ii) Range on horizontal ground**

If the ground is horizontal, the time at which the projectile lands is given by putting $y = 0$ in the equation $y = u\sin\theta\, t - \frac{1}{2}gt^2$. This gives $t = \frac{2u\sin\theta}{g}$. This is called the **time of flight**. If at this time the $x$-coordinate is $r$, then

$$r = u\cos\theta\, t = \frac{u\cos\theta \times 2u\sin\theta}{g} = \frac{2u^2\sin\theta\cos\theta}{g}.$$

It is shown in P2&3 Section 5.5 that $2\sin\theta\cos\theta$ is the expanded form of $\sin 2\theta$. So you can write the formula more simply as

$$r = \frac{u^2\sin 2\theta}{g}.$$

**(iii) Maximum range on horizontal ground**

Suppose that the initial speed $u$ is known, but that $\theta$ can vary. You will see from the graph of $\sin 2\theta$ (Fig. 1.10) that $r$ takes its greatest value when $\theta = 45°$, and that $r_{max} = \frac{u^2}{g}$.

Also, from the symmetry of the graph it follows that $r$ has the same value when $\theta = \alpha°$ and when $\theta = (90 - \alpha)°$. So any point closer than the maximum range can be reached by either of two trajectories, one with a low angle of projection ($\alpha < 45$) and one with a high angle ($\beta = 90 - \alpha > 45$).

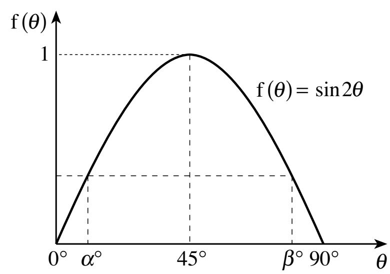



Fig. 1.10

**(iv) Equation of the trajectory**

You can think of the equations for $x$ and $y$ in terms of $t$ given in Section 1.2, page 6, as parametric equations for the trajectory, using time as the parameter. The cartesian equation can be found by turning $x = u\cos\theta\, t$ round to give $t = \frac{x}{u\cos\theta}$, and then substituting for $t$ in the equation for $y$:

$$y = u\sin\theta \times \frac{x}{u\cos\theta} - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2.$$

You can write this more simply by replacing $\frac{\sin\theta}{\cos\theta}$ by $\tan\theta$. Then

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}.$$

Notice that, since $u$, $g$ and $\theta$ are constant, this equation has the form $y = ax - bx^2$. You know that this is a parabola, and it is not difficult to show by differentiation that its vertex has coordinates $\left(\frac{a}{2b}, \frac{a^2}{4b}\right)$.

Writing $a = \tan\theta$ and $b = \frac{g}{2u^2\cos^2\theta}$, the vertex becomes $\left(\frac{u^2\sin 2\theta}{2g}, \frac{u^2\sin^2\theta}{2g}\right)$. This is another way of finding the formulae for the range and the greatest height.

*Check the details for yourself.*

For a projectile having initial velocity of magnitude $u$ at an angle $\theta$ to the horizontal, under gravity but neglecting air resistance:

the greatest height reached is $\frac{u^2\sin^2\theta}{2g}$;

the time to return to its original height is $\frac{2u\sin\theta}{g}$;

the range on horizontal ground is $\frac{u^2\sin 2\theta}{g}$;

the maximum range on horizontal ground is $\frac{u^2}{g}$;

the equation of the trajectory is

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}, \quad \text{or} \quad y = x\tan\theta - \frac{gx^2\sec^2\theta}{2u^2}.$$

**Example 1.3.1**

A basketball player throws the ball into the net, which is 3 metres horizontally from and 1 metre above the player's hands. The ball is thrown at $50°$ to the horizontal. How fast is it thrown?

Taking the player's hands as origin, you are given that $y = 1$ when $x = 3$ and that $\theta = 50°$. If you substitute these numbers into the equation of the trajectory you get

$$1 = 3\tan 50° - \frac{10 \times 9}{2u^2\cos^2 50°}.$$

This gives

$$\frac{45}{u^2\cos^2 50°} = 3\tan 50° - 1 = 2.575\ldots,$$

$$u^2 = \frac{45}{\cos^2 50^\circ \times 2.575\ldots} = 42.29\ldots,$$

$u = 6.50$, to 3 significant figures.

The ball is thrown with a speed of about $6.5\ \mathrm{m\,s^{-1}}$.

**Example 1.3.2**
A boy is standing on the beach and his sister is at the top of a cliff $6$ metres away at a height of 3 metres. He throws her an apple with a speed of $10\ \mathrm{m\,s^{-1}}$. In what direction should he throw it?

You are given that $y = 3$ when $x = 6$ and that $u = 10$. It is more convenient to use the second form of the equation of the trajectory. Substituting the given numbers,

$$3 = 6\tan\theta - \frac{10 \times 6^2 \times \sec^2\theta}{2 \times 10^2},$$

which simplifies to

$$3\sec^2\theta - 10\tan\theta + 5 = 0.$$

To solve this equation you can use the identity $\sec^2\theta \equiv 1 + \tan^2\theta$. Then

$$3\left(1 + \tan^2\theta\right) - 10\tan\theta + 5 = 0, \quad \text{that is} \quad 3\tan^2\theta - 10\tan\theta + 8 = 0.$$

This is a quadratic equation for $\tan\theta$, which factorises as

$$(3\tan\theta - 4)(\tan\theta - 2) = 0, \quad \text{so} \quad \tan\theta = \tfrac{4}{3} \text{ or } \tan\theta = 2.$$

The apple should be thrown at either $\tan^{-1}\frac{4}{3}$ or $\tan^{-1} 2$ to the horizontal, that is either $53.1^\circ$ or $63.4^\circ$.

## 1.4* Accessible points

If you launch a projectile from $O$ with a given initial speed $u$, but in an unspecified direction, you can obviously reach the points close to $O$, but not all points further away. You can use the method in Example 1.3.2 to find which points can be reached.

In Example 1.3.2 numerical values were given for $x$, $y$ and $u$. If instead you keep these in algebraic form, then the equation of the trajectory can be written as

$$y = x\tan\theta - \frac{gx^2\left(1 + \tan^2\theta\right)}{2u^2}.$$

This can then be arranged as a quadratic equation for $\tan\theta$,

$$gx^2\tan^2\theta - 2u^2x\tan\theta + \left(gx^2 + 2u^2y\right) = 0.$$

Now this equation can be solved to give values for $\tan\theta$ provided that the discriminant (that is, $b^2 - 4ac$ in the usual notation for quadratics) is greater than or equal to $0$. For this equation, the condition is

$$4u^4x^2 - 4gx^2\left(gx^2 + 2u^2y\right) \geqslant 0.$$

After cancelling $4x^2$, this can be rearranged as

$$y \leqslant \frac{u^2}{2g} - \frac{gx^2}{2u^2}.$$

Suppose, for example, that the initial speed is $10\text{ m s}^{-1}$. Then, in metre units, with $g = 10$, $\frac{u^2}{g} = 10$, so this inequality becomes

$$y \leqslant 5 - \tfrac{1}{20}x^2.$$

This is illustrated in Fig. 1.11, which shows several possible trajectories with this initial speed for various angles $\theta$. All the points on these curves lie on or underneath the parabola with equation $y = 5 - \frac{1}{20}x^2$. This is called the **bounding parabola** for this initial speed. It separates the points which can be reached from $O$ from those which can't.

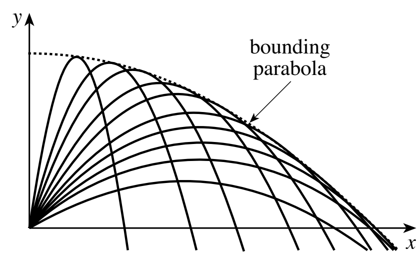



Fig. 1.11

Thus in Example 1.3.2 the coordinates of the girl were $(6,3)$. Since $5 - \frac{1}{20} \times 6^2 = 3.2 > 3$, these coordinates satisfy the inequality, so it is possible for her brother to throw the apple to reach her.

## Exercise 1C

Assume that all motion takes place above horizontal ground unless otherwise stated.

**1** A golfer strikes the ball with speed $60\text{ m s}^{-1}$. The ball lands in a bunker at the same level 210 metres away. Calculate the possible angles of projection.

**2** A projectile is launched at $45°$ to the horizontal. It lands 1.28 km from the point of projection. Calculate the initial speed.

**3** A footballer taking a free kick projects the ball with a speed of $20\text{ m s}^{-1}$ at $40°$ to the horizontal. Calculate the time of flight of the ball. How far from the point of the free kick would the ball hit the ground?

**4** A stone being skimmed across the surface of a lake makes an angle of $15°$ with the horizontal as it leaves the surface of the water, and remains in the air for $0.6$ seconds before its next bounce. Calculate the speed of the stone when it leaves the surface of the lake.

**5** A projectile launched with speed $75 \text{ m s}^{-1}$ is in the air for 14 seconds. Calculate the angle of projection.

**6** An astronaut who can drive a golf ball a maximum distance of 350 metres on Earth can drive it 430 metres on planet Zog. Calculate the acceleration due to gravity on Zog.

**7** An archer releases an arrow with speed $70 \text{ m s}^{-1}$ at an angle of $25°$ to the horizontal. Calculate the range of the arrow. Determine the height of the arrow above its initial level when it has travelled a horizontal distance of 50 metres, and find the other horizontal distance for which it has the same height.

**8** A particle projected at an angle of $40°$ passes through the point with coordinates $(70,28)$ metres. Find the initial speed of the particle.

**9** A hockey player taking a free hit projects the ball with speed $12.5 \text{ m s}^{-1}$. A player 10 metres away intercepts the ball at a height of 1.8 metres. Calculate the angle of projection.

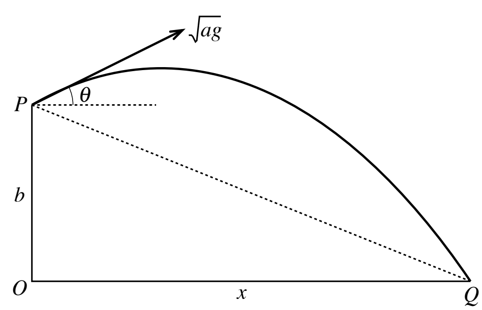
**10** The equation of the path of a projectile is $y = 0.5x - 0.02x^2$. Determine the initial speed of the projectile.

**11** A tennis player strikes the ball at a height of 0.5 metres. It passes above her opponent 10 metres away at a height of 4 metres, and lands 20 metres from the first player, who has not moved since striking the ball. Calculate the angle of projection of the ball.

**12** In a game of cricket, a batsman strikes the ball at a height of 1 metre. It passes over a fielder 7 metres from the bat at a height of 3 metres, and hits the ground 60 metres from the bat. How fast was the ball hit?

**13** The greatest height reached by a projectile is one-tenth of its range on horizontal ground. Calculate the angle of projection.

**14** A soldier at position $P$ fires a mortar shell with speed $\sqrt{ag}$ at an angle $\theta$ above the horizontal, where $a$ is a constant. $P$ is at a vertical height $b$ above a horizontal plane. The shell strikes the plane at the point $Q$, and $O$ is the point at the level of the plane vertically below $P$, as shown in the diagram. Letting $OQ = x$, obtain the equation




$$x^2 \tan^2\theta - 2ax\tan\theta + \left(x^2 - 2ab\right) = 0.$$

Show that the maximum value of $x$, as $\theta$ varies, is $\sqrt{a(a+2b)}$ and that this is achieved when $\tan\theta = \sqrt{\dfrac{a}{a+2b}}$.

The sound of the shell being fired travels along the straight line $PQ$ at a constant speed $\sqrt{cg}$. Given that the shell is fired to achieve its maximum range, show that if a man standing at $Q$ hears the sound of firing before the shell arrives at $Q$, so giving him time to take cover, then $c > \frac{1}{2}(a+b)$. (OCR)

## Miscellaneous exercise 1

**1** A girl throws a stone which breaks a window 2 seconds later. The speed of projection is $20 \text{ m s}^{-1}$ and the angle of projection is $60°$. Assuming that the motion can be modelled by a particle moving with constant acceleration, find the horizontal and vertical components of the velocity of the stone just before impact. (OCR)

**2** A ball is projected from a point on horizontal ground. The speed of projection is $30 \text{ m s}^{-1}$ and the greatest height reached is 20 metres. Assuming no air resistance, find the angle of projection above the horizontal and the speed of the ball as it passes through the highest point. (OCR)

**3** A ball is thrown with speed $28 \text{ m s}^{-1}$ at an angle of $40°$ above the horizontal. After 3 seconds the ball is at $P$. Ignoring air resistance, find the magnitude and direction of the velocity of the ball as it passes through $P$. (OCR)

**4** A particle is projected at a speed of $45 \text{ m s}^{-1}$ at an angle of $30°$ above the horizontal.

(a) Calculate

(i) the speed and direction of motion of the particle after 4 seconds,

(ii) the maximum height, above the point of projection, reached by the particle.

(b) State one assumption made in modelling the motion of a particle. (OCR)

**5** A ball is projected from the point $O$, with velocity $\begin{pmatrix} u \\ v \end{pmatrix}$ $\text{m s}^{-1}$. The highest point reached in the subsequent motion is 30 metres above the level of $O$, and at the highest point the speed of the ball is $20 \text{ m s}^{-1}$. Assuming the ball moves with constant acceleration, find the values of $u$ and $v$. The ball passes through the point $S$ which is above the level of $O$ and 80 metres horizontally from $O$. Find the direction of motion of the ball at $S$. (OCR)

**6** An athlete throws a heavy ball. The ball is released with an initial speed of $10.3 \text{ m s}^{-1}$ at an angle of $35°$ above the horizontal. The point of release is 1.68 metres above the ground.

(a) Neglecting air resistance show that the equation of the path of the ball is approximately $y = 0.7x - 0.07x^2$, where $y$ metres is the height of the ball above the release point when it has travelled a horizontal distance of $x$ metres.

(b) Find how far the ball has been thrown horizontally before it strikes the horizontal, flat ground.

(c) State one further aspect of the motion of the ball which is not incorporated in the model. (OCR)

**7** A fielder can throw a cricket ball faster at low angles than at high angles. This is modelled by assuming that, at an angle $\theta$, he can throw a ball with a speed $k\sqrt{\cos\theta}$, where $k$ is a constant.

(a) Show that the horizontal distance he can throw is given by $\dfrac{2k^2}{g}\left(\sin\theta - \sin^3\theta\right)$.

(b) Find the maximum distance he can throw the ball on level ground.

**8** A boy is trying to throw a pellet of bread to a young bird sitting on a ledge in a wall, at a height of 11 metres above the ground. The boy is standing 5 metres away from the wall and throws the pellet from a height of 1.5 metres above the ground. When the pellet reaches the ledge it is moving horizontally. Treating the pellet as a particle moving with constant acceleration, find

(a) the vertical component of the initial velocity,

(b) the time taken for the pellet to reach the ledge,

(c) the speed of the pellet when it reaches the ledge.

State a force that has been neglected in the above model. (OCR)

**9** (a) At a point $O$ on its path, a projectile has speed $V$ and is travelling at an angle $\alpha$ above the horizontal. Derive the equation of the trajectory of the projectile in the form

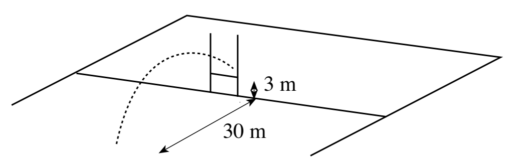
$$y = x\tan\alpha - \frac{gx^2}{2V^2}\left(1+\tan^2\alpha\right),$$

where the $x$- and $y$-axes pass through $O$ and are directed horizontally and vertically upwards respectively, and state briefly one simplifying assumption that is necessary in obtaining this result.

(b) An aircraft is flown on a path given by $y = 0.28x - \left(2.4\times10^{-4}\right)x^2$, in such a way that the acceleration of the aircraft has the constant value $10\text{ m s}^{-2}$ vertically downwards. The units of $x$ and $y$ are metres. By comparing the equation of the aircraft's path with the standard trajectory equation in (a), find the speed and direction of motion of the aircraft at the point $(0,0)$. Calculate the time of flight between the two points on the aircraft's path at which $y = 0$. (OCR, adapted)

(By flying an aircraft on a parabolic path with appropriate speed, it is possible to simulate within the fuselage a weightless environment. This has been used for astronaut training.)

**10** A player in a rugby match kicks the ball from the ground over a crossbar which is at a height of 3 metres above the ground and at a horizontal distance of 30 metres from the ball as shown in the diagram. The player kicks the ball so that it leaves the ground at an angle of $25°$ above the horizontal.




(a) Find the minimum speed at which the ball must be kicked.

In fact the player kicked the ball with an initial speed of $25\text{ m s}^{-1}$.

(b) (i) Find the time that elapsed between the ball leaving the ground and it passing vertically above the crossbar.

(ii) Find the angle to the horizontal made by the direction of motion of the ball as it passed over the crossbar. You should make it clear whether the ball is ascending or descending at this time.

(c) State two assumptions that you have made in modelling the motion of the ball which may not be reasonable in practice. (OCR)

**11** A particle $P$ is projected with speed $10.1\,\mathrm{m\,s^{-1}}$ from a point $O$ on horizontal ground. The angle of projection is $\alpha$ above the horizontal. At time $t$ seconds after the instant of projection, the horizontal distance travelled by $P$ is $x$ metres and the height of $P$ above the ground is $y$ metres. Neglecting air resistance, write down expressions for $x$ and $y$ in terms of $\alpha$ and $t$. Hence

(a) show that $P$ reaches the ground when $t = 2.02\sin\alpha$, approximately,

(b) find, in terms of $\alpha$, the value of $x$ when $P$ reaches the ground,

(c) state the value of $t$, in terms of $\alpha$, for which $P$ is at its maximum height and show that the maximum height of $P$ above the ground is $5.10\sin^2\alpha$ metres, approximately,

(d) find an expression for $y$ in terms of $x$ and $\alpha$.

Given that $\alpha = 45°$, use your answer to part (d) to find the horizontal distance travelled by $P$ when this horizontal distance is 8 times the height of $P$ above the ground.

(OCR, adapted)

**12** A stone is projected from the point $O$ with speed $20\ \mathrm{m\,s^{-1}}$ at an angle $\theta$ above the horizontal, where $\sin\theta = \frac{3}{5}$. The point $A$ is on horizontal ground and is 30 metres directly below $O$. The stone hits the ground at the point $B$. Modelling the motion of the stone as that of a particle moving with constant acceleration, find

(a) the distance $AB$,

(b) the distance of the stone below the level of $O$ at the instant that its direction of motion makes an angle of $50°$ with the horizontal.

$T$ seconds after projection, the stone is at the point $R$, where angle $RAB = 20°$. Find the value of $T$.

(OCR)