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Mathematics

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Contents

MODULE 1 Working with numbers

We all work confidently with numbers every day, often without really thinking about them. Professor Brian Butterworth, in his book *The Mathematical Brain*, estimates that he processes about 1 000 numbers per hour – that's about 6 million numbers a year – without doing anything special! Butterworth is not special, nor is he a mathematician. The numbers he processes are all found in daily life – 51 numbers on the first page of a newspaper (prices, dates, amounts); numbers on the radio (stations, frequencies, news, hit parades); sporting results; time; labels on food; instructions for cooking; money; addresses; car registration plates; bar codes; page numbers in a book, and many, many more examples. What is more, these numbers are different: some are whole numbers, some are fractions, some are decimals, some are in order, some are random. His point is simple – we depend on numbers every day, and we need to understand number systems and how to use them.

Remember

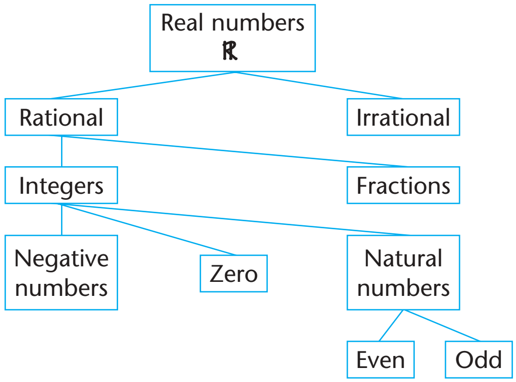
When a list of numbers is followed by … it means that the list does not end with the last number written. It can continue infinitely.

Number systems

Different sets of numbers can be defined in the following ways:

Natural numbers ($\mathbb{N}$)	1, 2, 3, 4, 5, …
Whole numbers	0, 1, 2, 3, 4, 5, …
Integers (also called directed numbers) • Positive integers • Negative integers	…, –5, –4, –3, –2, –1, 0, 1, 2, 3, 4, 5, … 1, 2, 3, 4, 5, … (positive numbers are written without a + sign) –1, –2, –3, –4, –5, …
Even numbers	2, 4, 6, 8, 10, … (can be divided by 2)
Odd numbers	1, 3, 5, 7, 9, … (not divisible by 2)
Prime numbers – natural numbers that are divisible by 1 and the number itself – these numbers have only two factors	2, 3, 5, 7, 11, 13, … 1 is not a prime number as it only has one factor 2 is the only even prime number The largest known prime has thousands of digits
Square numbers – when natural numbers are multiplied by themselves you get a square number	1, 4, 9, 16, 25, 36, …
Fractions – these are parts of whole numbers, also called rational numbers	$\frac{1}{2}$, $\frac{1}{4}$, $2\frac{1}{2}$ (these are vulgar or common fractions; the top line is called numerator, bottom line is called denominator) 0.5, 0.4, 0.335 (these are decimal fractions) Percentages (%) are fractions of 100

These numbers all fit into the larger set of real numbers ($\mathbb{R}$). You can see how they fit on the diagram on the next page.



Mathematical symbols

In addition to numbers, symbols are used to write mathematical information. Make sure you recognise the following symbols:

+ Plus or positive	= Equal to	< Less than	∴ Therefore
– Subtract or minus	≠ Not equal to	> Greater than	∞ Infinity
× Multiply	≈ Approximately equal to	≤ Less than or equal to	∥ Parallel to
÷ Divide	: Is to / such that	≥ Greater than or equal to	√ Square root

Exercise

1. Which of these numbers are natural numbers?
   3, –2, 0, 1, 15, 4, 5
2. Which of the following are integers?
   –7, 10, 32, –32, 0
3. Which of these are prime numbers?
   21, 23, 25, 27, 29, 31
4. Write down the next two prime numbers after:
   a) 30 b) 80.
5. Are all prime numbers odd?
6. Are all odd numbers prime?
7. If you add two odd numbers, what kind of number do you get?
8. What kind of number will you get when you add an odd and an even number?
9. Write down:
   a) four square numbers bigger than 25
   b) four rational numbers smaller than $\frac{1}{2}$
   c) the set of negative integers between –7 and 0
   d) the decimal equivalents of $\frac{1}{2}$, 75% and $1\frac{1}{2}$.

Using a calculator

The table shows you some of the functions that are found on most calculators. However, all calculators are different and you must read the instruction manual and make sure you know how to use the model you have.

Key	Function	Key	Function
INV or 2ndF	Used to access some functions	EXP or EE	This is the standard form button
C	Cancels the last number entered	$a^{b}/_{c}$	This is the fraction key; to enter $\frac{2}{3}$, press 2 $a^{b}/_{c}$ 3
AC	Cancels all data entered	Min or STO	Stores the displayed number in memory
x^2	Calculates the square of a number	MR or RCL	Recalls whatever is in the memory
x^3	Calculates the cube of a number	M+	Adds the display value to memory
$\sqrt{\ }$	Calculates the square root of a number	M–	Subtracts the display value from memory
$\sqrt[3]{\ }$	Calculates the cube root of a number	Mode	Gives the mode for calculation (refer to your manual)
+/–	Reverses the sign (changes positive to negative or negative to positive)	DRG	Changes units to degrees, radians or grads; your calculator should be normally set in degrees
x^y	Calculates any power; to calculate 2^4, you key in 2 x^y 4	sin cos tan	Calculates sine, cosine or tangent values (trigonometry)

Exercise

Use your calculator to work out:

1. 5^2
2. $\sqrt{2.6}$
3. $\frac{3}{4} - \frac{1}{5}$
4. $(3 \times 10^7) - (2 \times 10^6)$
5. 3^7
6. $-7 + 3 \div 2.$

Calculation rules

Mathematics is governed by a set of rules that has been developed to avoid confusion when working with complicated operations. These rules tell you about the order of operations – in other words, what you should do first in a sum like this one: $3 \times 4 + 14 \div 2$.

One way to remember the order of operations is to use a system called BODMAS. This is a mnemonic that stands for:

B – Brackets. Work out anything in brackets first. When there is more than one set of brackets, work from the outside to the inner sets.
O – Of. Change 'of' to '×' and work it out.
D – Divide.
M – Multiply. When there are only × and ÷ signs in a sum, you can work in any order.
A – Add.
S – Subtract. When there are only + and – signs in a sum, you can work in any order.

Remember

The sign in front of the number *belongs* to that number. So if you change the order of the numbers, remember to move their signs with them.

Most modern calculators are programmed to follow BODMAS rules.

These examples can help you to see how BODMAS works:

Examples

B
O
D
M
A
S

1. $3 \times 4 + 14 \div 2$ (÷)
$= 3 \times 4 + 7$ (×)
$= 12 + 7$ (+)
$= 19$

2. $18 - 14 \div (3 + 4) + 2 \times 3$ (Brackets)
$= 18 - 14 \div 7 + 2 \times 3$ (÷ and ×)
$= 18 - 2 + 6$ (– and +)
$= 22$

Exercise

Evaluate:

1. $(16 - 10) \div 2$
2. $16 - 10 \div 2$
3. $(4 + 3) \times 2$
4. $4 + 3 \times 2$
5. $(14 - 5) \div (20 - 2)$
6. $30 + 132 \div 11$
7. $5 \times 5 + 6 \div 2$
8. $2 + 5 \div 3 \times 6.$

Fill in signs to make these expressions true:

9. $5 \square 3$
10. $-5 \square -3$
11. $5.7 \square \frac{5}{7}$
12. $2\frac{1}{2} \square 2.5$
13. $4 = \boxed{16}$
14. $\frac{1}{3} \square 0.333$
15. $3.14 \square \frac{22}{7}$
16. $-7 \square \frac{1}{2}$
17. $3.333 \square \frac{1}{3}$
18. $-2^2 \square -4$

Factors

A *factor* of a number is a number that will divide exactly into it.

Consider the number 12.
The factors of 12 are: 1, 2, 3, 4, 6 and 12. That is, 1, 2, 3, 4, 6 and 12 will divide exactly into 12.

Of these factors, 2 and 3 are called the *prime factors*, because they are also prime numbers.

Prime factors

Prime factors of a number are factors of the number that are prime numbers.

You can write every number as the product of two or more numbers. For example, $12 = 4 \times 3$. But notice that 4 is not a prime number. You can break 4 down further. So now you have $12 = 2 \times 2 \times 3$. Since 2 and 3 are both prime numbers, you can say that you've written 12 as the *product of prime factors*.

Writing numbers as the product of prime factors

To write a number as a product of its prime factors, first try to divide the given number by the first prime number, 2. Continue until 2 will no longer divide into it exactly. Then try the next prime number, which is 3, then 5 and so on, until the final answer is 1.

Hint

To find the product of two or more numbers means you must *multiply* those numbers together.

Examples

1. Write 60 as a product of prime factors.

Divide 60 by 2 as shown	2	60
Again, divide by 2	2	30
Now, you can divide 15 by 3	3	15
Finally, divide by 5	5	5
		1

So $60 = 2 \times 2 \times 3 \times 5$

2. Write 1 617 as a product of prime factors.
1 617 cannot be divided exactly by 2, but it will divide exactly by the next prime number, which is 3.

539 cannot be divided exactly by 3	3	1 617
or 5 but it will divide exactly by 7	7	539
Divide by 7 again	7	77
Finally, divide by 11	11	11
		1

So $1\,617 = 3 \times 7 \times 7 \times 11$

Multiples

A *multiple* of a number is the product of that number and an integer. Multiples of 3 are 3, 6, 9, 12, … How do you find them?

$3 \times 1 = 3$, $3 \times 2 = 6$, $3 \times 3 = 9$ …

So 6 is a *multiple* of 3 because $6 = 3 \times 2$. Also, 3 is a *factor* of 6 because $6 \div 3 = 2$.

Lowest common multiple (LCM)

The *lowest common multiple* (LCM) of two or more numbers is the smallest number that is a multiple of each of them. 12 is a multiple of 3 and 4. It is also the smallest number that *both* 3 and 4 will divide into. So 12 is the LCM of 3 and 4.

There are two methods of finding the LCM of numbers. The first method is to list the multiples of each number and then pick out the lowest number that appears in every one of the lists. That is, the *lowest* number *common* to all the lists.

The second method is to express each of the numbers as a product of prime factors and then work out the smallest number which includes each of these products.

Some examples will make these methods clearer to you.

Example

Find the LCM of 12 and 15.

Method A: The multiples of 12 are 12, 24, 36, 48, (60), 72, …
The multiples of 15 are 15, 30, 45, (60), 75, 90, …
The lowest number that appears in both these lists is 60.

So the LCM of 12 and 15 is 60.

Method B: Expressing each number as a product of prime numbers:
$12 = 2 \times 2 \times 3$
$15 = 3 \times 5$

Any number that is a multiple of 12 must have at least two 2s and one 3 in its prime factor form.
Any number that is a multiple of 15 must have at least one 3 and one 5 in its prime factor form.
So any common multiple of 12 and 15 must have at least two 2s, one 3 and one 5 in its prime factor form.
The lowest common multiple of 12 and 15 is $2 \times 2 \times 3 \times 5$.

So the LCM of 12 and 15 is 60.

Exercise

1. Write the following numbers as a product of prime factors.

   a) 18 b) 16 c) 64 d) 81 e) 100
   f) 36 g) 21 h) 11 i) 45 j) 108

In each of the following questions, find the LCM of the given numbers.

2. a) 9 and 12 b) 12 and 18 c) 15 and 24
   d) 24 and 36 e) 3 and 5

3. a) 4, 14 and 21 b) 4, 9 and 18 c) 12, 16 and 24
   d) 6, 10 and 15

Patterns and sequences

Look at the following patterns of numbers.
1, 2, 3, 4, 5, …
5, 10, 15, 20, 25, …
What are the next three numbers in each pattern?

6, 7, 8 are the next three numbers in the first pattern and 30, 35, 40 are the next three numbers in the second pattern.

There is a special name for such patterns of numbers. They are called *sequences*. Each number in a sequence is called a *term* of the sequence. Each sequence is a group of numbers that has two important properties. The numbers are listed in a particular order and there is a rule that enables you to continue the sequence. Either the rule is given, or the first few terms of the sequence are given and you have to work out the rule.

Examples

Write down the next two terms in each of the following sequences:

1. 2, 6, 10, 14, 18, …
2. 2, 6, 18, 54, 162, …
3. 27, 22, 17, 12, 7, …
4. 2, 3, 5, 8, 12, 17, …

1. In the sequence, each term is 4 more than the previous term. The next two terms are 22 (that is 18 + 4) and 26.
2. In the sequence, each term is 3 times the previous term. The next two terms are 486 (that is 162×3) and 1 458.
3. In the sequence, each term is 5 less than the previous term. The next two terms are 2 (that is 7 – 5) and –3.
4. The increases from one term to the next are 1, 2, 3, 4, 5. The next two increases will be 6 and 7, so the next two terms are 23 (that is 17 + 6) and 30 (that is 23 + 7).

Exercise

1. Write down the next two terms in each of the following sequences:
   a) 2, 4, 6, 8, …
   b) 3, 6, 9, 12, …
   c) 2, 3, 5, 7, 11, 13, …
   d) 9, 6, 3, 0, –3, …
   e) 3, 4, 6, 9, 13, …
   f) 1, 3, 4, 7, 11, 18, 29, …
2. Find two square numbers, each less than 100, which are also cube numbers.

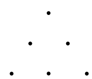
Hint

Consecutive means that the numbers are next to one another in the sequence.

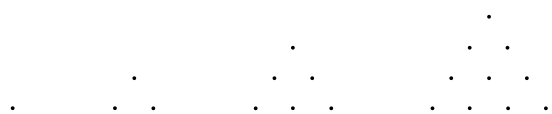
3. Study the sums of consecutive odd numbers shown below.

$$1 = 1$$
$$1 + 3 = 4$$
$$1 + 3 + 5 = 9$$
$$1 + 3 + 5 + 7 = 16$$
$$1 + 3 + 5 + 7 + 9 = 25$$

   a) Write down the next two lines of this pattern.
   b) Work out the sum of the first 10 odd numbers.
   c) Work out the sum of the first 210 odd numbers.
   d) Try to write a general rule for finding the sum of *n* consecutive odd numbers.

4. Six dots can be arranged in a triangle  so 6 is a *triangle number*.

   The sequence of triangle numbers starts with 1, 3, 6, 10, …



   a) Write down the next two triangle numbers.
   b) Study the sums of pairs of consecutive triangle numbers.
      1 + 3 = 4
      3 + 6 = 9
      6 + 10 = 16
      (i) Write down the next two lines of this pattern.
      (ii) What special type of numbers are these sums of consecutive triangle numbers?
      (iii) Work out the sum of the 10th triangle number and the 11th triangle number.
5. In each of the following, one number must be removed or replaced to make the sequence work. Work out which number this is and rewrite the sequences.
   a) 1, 7, 13, 16, 25, 31, …
   b) 1, 4, 7, 9, 16, 25, …

Sets

A set is a well-defined collection of objects that usually have some connection with each other. Sets can be described in words. For example: set A is a set of the oceans of the world; set B contains natural numbers less than or equal to 10. Sets can also be listed between curly brackets { } or braces. For example:

A = {Indian, Atlantic, Pacific, Arctic, Antarctic}

B = {1, 2, 3, 4, 5, 6, 7, 8, 9}

Objects that belong to a set are called elements and are indicated by the symbol $\in$. $\in$ means 'is an element of'. In the examples above, we can say Atlantic $\in$ A or 2 $\in$ B.

$\notin$ means 'is not an element of'. Again using our examples, we can say Mount Everest $\notin$ A and 11 $\notin$ B.

The sets described and listed above are finite sets – they have a fixed number of elements.

Sets that do not have a fixed number of elements are infinite sets. The set of natural numbers greater than 10 is an example of an infinite set. This can be listed as {10, 11, 12, 13, …}.

A set may also have no elements. Such a set is called an empty set. The symbols { } or $\varnothing$ indicate an empty set. An example of an empty set would be women over 6 m tall, or square circles. The number of elements in an empty set is 0 but {0} is not an empty set – it is a set containing one element, 0.

Set builder notation

This is a method of defining a set when it is difficult to list all the elements or when the elements are unknown.

Examples

1. The infinite set of even natural numbers can be listed like this: {2; 4; 6; 8; 10; ...}. This set can be described in set builder notation like this: $\{x: x \in \mathbb{N}, x \text{ is an even number}\}$. You read this as 'the set of all elements x such that x is an element of the set of natural numbers and is even'.
2. The finite set of natural numbers between 9 and 90 can be listed as {10; 11; 12; 13; …; 89} or written in set builder notation as $\{x: x \in \mathbb{N}, 9 < x < 90\}$. You read this as 'the set of elements x such that x is an element of the set of natural numbers and x is bigger than 9 and smaller than 90'.

Exercise

1. List these sets.
   a) The set of persons living in your home.
   b) The set consisting of the first five odd numbers.
   c) The set whose objects are the last four letters of the English alphabet.
   d) The set of even numbers between 1 and 7.

2. Describe these sets.
   a) $A = \{2, 3, 5, 7\}$
   b) $P = \{s, t, u, v, w, x, y, z\}$
   c) $Q = \{5, 10, 15, 20, 25\}$
3. $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 6, 8, 10\}$, $C = \{3, 5, 7, 9, 11\}$.
   List the sets given by:
   a) $\{x: x \in A, x > 3\}$
   b) $\{x: x \in B, x \leq 6\}$
   c) $\{x: x \in C, 5 < x < 12\}$
   d) $\{x: x = 2y + 1, y \in B\}$
   e) $\{(x, y): x \in B, y \in C, x = 2y\}$.
4. Express in set builder notation the set of $\mathbb{N}$:
   a) greater than 5
   b) less than 10
   c) between 3 and 11
   d) which are prime.

Relationships between sets

Equal sets

Sets that contain exactly the same elements are said to be equal.
Consider these sets:

A = The set of letters in the word END = {E, N, D}
B = The set of letters in the word DEN = {D, E, N}

We can say A = B.
The order of the elements in the set does not matter.

Subsets

If every element of set A is also an element of set B, then A is a subset of B. This is written as $A \subset B$. $\subset$ means 'is a proper subset of'. $\not\subset$ means 'not a proper subset of'. Proper subsets always contain fewer elements than the set itself.

The proper subsets of {D, E, N} are:
{D} {E} {N} {D, E} {D, N} {E, N}
Trivial subsets of {D, E, N} are { } (the empty set) and {D, E, N}.

If a set has n elements, it will have 2^n subsets. For example, a set with 3 elements will have 2^3 subsets. That is $2 \times 2 \times 2 = 8$ subsets.

The set of elements from which to select to form subsets is called the universal set. The symbol $\mathscr{E}$ is used to denote the universal set. You should remember that the universal set could change from problem to problem.

Exercise

1. Find a universal set for each of the following sets.
   a) the set of people in your class that have long hair
   b) the set of vowels
   c) {2, 4, 6, 8}
   d) {goats, sheep, cattle}
2. If $\mathscr{E}$ is the set of students at your school, define five subsets of $\mathscr{E}$.

Intersection and union of sets

The intersection of two sets refers to elements that are found in both sets. For example:

A = {a, b, c, d}

B = {c, d, e, f}

The intersection of these two sets is {c, d}. You write this as A ∩ B.

Remember that A ∩ B = B ∩ A.

Remember

n(A) means the number of elements in set A. In the union of sets, when n(A) + n(B) = n(C) then A and B were disjoint sets. When n(A) + n(B) ≠ n(C) then the sets were not disjoint; in other words A ∩ B ≠ ∅.

When two sets have no elements in common, they are called disjoint sets. The intersection of disjoint sets is the empty set or ∅.

The elements of two or more sets can be combined to make a new set. This is called the union of the sets. For example:

A = {1, 2, 3, 4}

B = {4, 5, 6}

C = {1, 2, 3, 4, 5, 6}

Set C is the union of set A and set B. We write this as A ∪ B = C.

Exercise

1. For each of the following, list the set which is the intersection of the two sets.
   a) {1, 2, 3, 4, 5, 6} and {4, 5, 8, 9, 10}
   b) {a, b, c, d} and {w, x, y, z}
   c) {1, 2, 3, 4, 5, …} and {1, 3, 5, 7, …}
   d) {m, n, o, p, q} and {d, o, g}
2. ℰ = {1, 2, 3, 4, 5, 6}, A = {1, 2, 3, 4}, B = {3, 4, 5}, C = {5}.
   List the sets A ∩ B, B ∩ C, A ∩ C and ℰ ∩ B.
3. Write down the union of the following sets.
   a) A = {a, b, c} and B = {d, e, f}
   b) A = {x} and B = {y}
   c) P = {2, 4, 6, 8, 10} and Q = {5, 6, 3, 1}
   d) {0, 1, 2, 3} and {4, 5, 6, …}
4. State whether the following statements are true or false.
   a) P ∪ Q = Q ∪ P b) M ∪ M = 2M
   c) B ∪ ∅ = ∅ d) B ∪ B = B^2

The complement of a set

The complement of a set A is the set of those elements that are in the universal set but are not in A. The complement of set A is written as A′.

For example, if ℰ = {1, 2, 3, 4, 5} and A = {2, 4, 5}, then all the members of ℰ that are not in A make the subset {1, 3}. This subset is the complement of A, so A′ = {1, 3}.

A set and its complement are disjoint. A ∩ A′ = ∅.

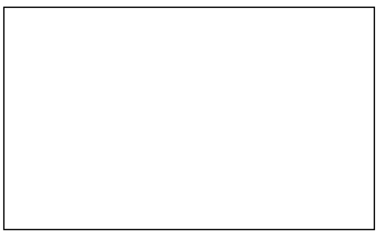
The union of a set and its complement is the universal set. A ∪ A′ = ℰ.

Exercise

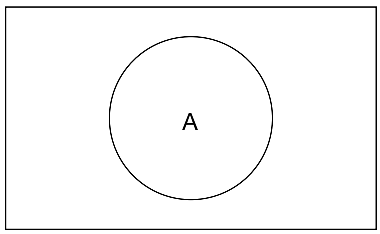
1. $\mathscr{E} = \{p, q, r, s, t, v\}$ and $Y = \{p, r, v\}$. List the members of Y'.
2. $\mathscr{E} = \{1, 2, 3, \ldots, 20\}$. P is the set of prime numbers in $\mathscr{E}$.
   a) List the elements of the set P.
   b) List the elements of the set P'.
3. W = {whole numbers} and subset T = {even numbers}.
   Describe in words the complement of T with respect to W.
4. The universal set is the set of all integers.
   What is the complement of the set of negative numbers?

Venn diagrams

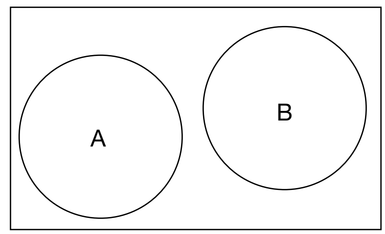
Sketches used to illustrate sets and the relationships between them are called Venn diagrams. You need to understand the basics of Venn diagrams before you can use them to help you solve problems involving sets.



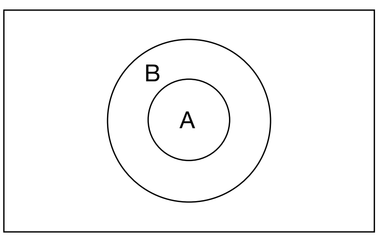
The rectangle represents $\mathscr{E}$



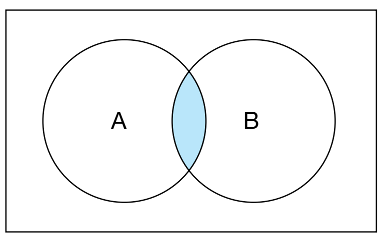
The circle represents set A



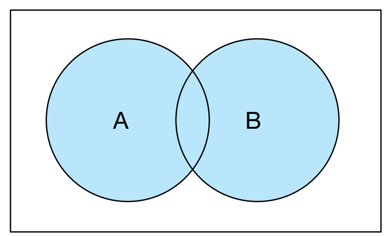
Set A and set B are disjoint



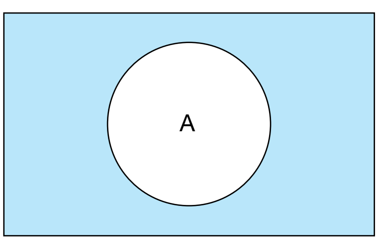
$A \subset B$



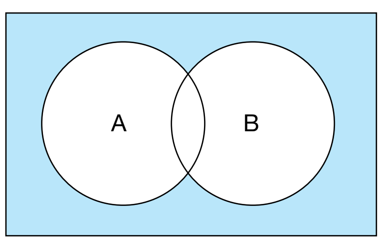
$A \cap B$ is the shaded portion



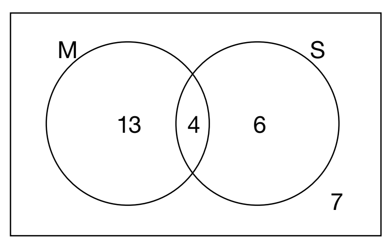
$A \cup B$ is the shaded portion



A' is the shaded portion



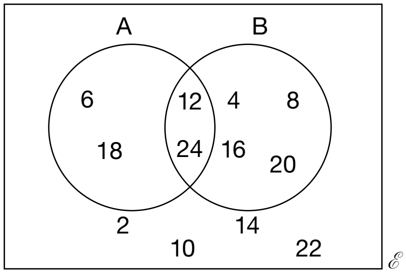
$(A \cup B)'$ is the shaded portion



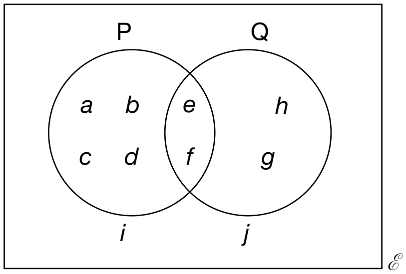
Venn diagrams can also be used to show the number of elements $n(A)$ in a set.
In this case:
M = {students doing maths}
S = {students doing science}

Exercise

1. Use the given Venn diagram to answer the following questions.
   a) List the elements of A and B.
   b) List the elements in A ∩ B.
   c) List the elements in A ∪ B.

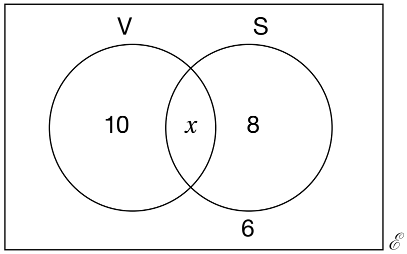


2. Use the given Venn diagram to answer the following questions.
   a) List the elements that belong to:
      (i) P
      (ii) Q.
   b) List the elements that belong to both P and Q.
   c) List the elements that belong to:
      (i) neither P nor Q
      (ii) P but not Q.

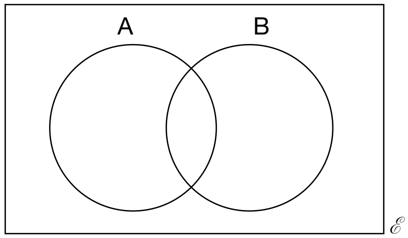


3. Draw a Venn diagram to show the following sets and write each element in its correct space.
   a) The universal set is {*a, b, c, d, e, f, g, h*}.
      A = {*b, c, f, g*} B = {*a, b, c, d, f*}
   b) ℰ = {whole numbers from 20 to 36 inclusive}
      A = {multiples of 4} B = {numbers greater than 29}

4. The universal set is {students in a class}.
   V = {students who like volleyball} and S = {students who play soccer}.
   There are 30 students in the class.
   The Venn diagram shows numbers of students.
   a) Find the value of x.
   b) How many students like volleyball?
   c) How many students in the class do not play soccer?



5. Shade the region in the Venn diagram which represents the subset A ∩ B′.



Powers and roots

Hint

Make sure you can find the correct buttons on your calculator to raise numbers to powers and to find the roots of powers.

Remember

- 3^2 is another way of writing 3×3.
- You write the square root of 64 as $\sqrt{64}$.
- 6^3 means $6 \times 6 \times 6$.
- $\sqrt[3]{216}$ means the cube root of 216.

When a number is multiplied by itself one or more times, the answer is a power of the first number. The number you started with is called a root of the power. For example: $5 \times 5 = 5^2 = 25$.

In this example, 5 is raised to the power of 2 or five is squared. The root of the power is 5. Because the root was squared, we say 5 is the *square root*.

Square numbers and roots

If a natural number is multiplied by itself, you get a *square* number. It may also be called a *perfect square*.

For example, $3 \times 3 = 3^2 = 9$ so 9 is a square number.

3 is said to be the *square root* of 9.

The first five square numbers are 1, 4, 9, 16 and 25.

Examples

1. $8 \times 8 = 64$ so 64 is a square number and its square root is 8. A shorter way of writing this is $8^2 = 64$ and $\sqrt{64} = 8$.
2. $12 \times 12 = 144$ so 144 is a square number and its square root is 12. Can you write this in a shorter way?

Cube numbers and roots

If the square of the natural number is multiplied again by the same natural number, the result is called a *cube number*.

For example, $6 \times 6 = 36$ and $36 \times 6 = 216$ so 216 is a cube number.

This is usually written as $6 \times 6 \times 6 = 216$ or $6^3 = 216$.

6 is said to be the *cube root* of 216.

The first five cube numbers are 1, 8, 27, 64 and 125.

Examples

1. $9 \times 9 \times 9 = 729$ so 729 is a cube number and its cube root is 9.
2. $20^3 = 8\,000$ so $\sqrt[3]{8\,000} = 20$

Exercise

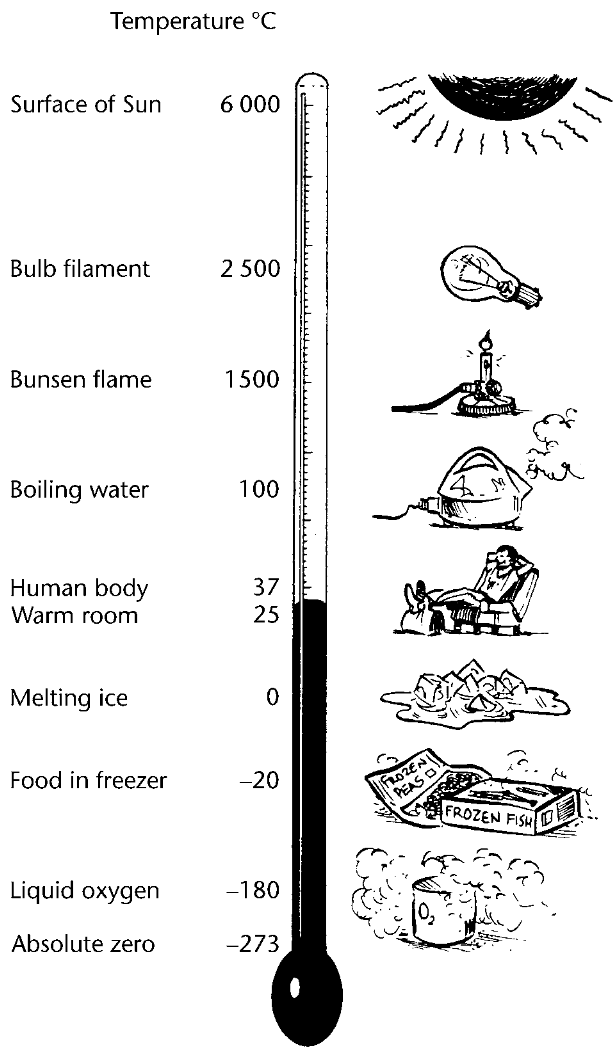
1. Add four more terms to each of these two sequences.
   a) 1, 4, 9, 16, 25, … b) 1, 8, 27, 64, 125, 216, …
2. Use your calculator to work out:
   a) $\sqrt{49}$; $\sqrt{121}$; $\sqrt{256}$; $\sqrt{8}$ b) $\sqrt[3]{125\,000}$; $\sqrt[3]{1\,728}$; $\sqrt[3]{729}$
3. Calculate:
   a) 13^2 b) 9^3
   c) $(-4)^2$ d) -3^3
   e) 100^2 f) 100^3.

Directed numbers

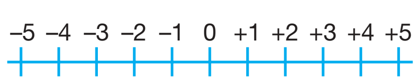
Look at the temperature of each object shown in the picture. Notice that some of the colder objects have temperatures that are below 0 °C. These are negative temperatures and are marked with a – sign. Notice also that temperatures above 0 °C are written without a + sign. In mathematics, we accept that numbers are positive unless they are marked as negative. Positive and negative numbers are also called directed numbers.

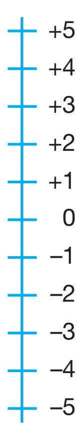
Remember

There are different systems of measuring temperature. We are using the Celsius scale, on which the freezing point of water is 0 °C and the boiling point of water is 100 °C. You may also find temperatures given in Fahrenheit. To convert temperatures from Celsius to Fahrenheit you can use the formula (°C × 1.8) + 32.



When you are working with directed numbers, it is helpful to represent them on a number line. Your number line can be either horizontal or vertical.





The number line can be used to add or subtract directed numbers. Find your starting position and then move left or right or up or down. The sign of the number indicates how you should move on the number line. +4 means four places to the right (or up). –4 means four places to the left (or down).

Examples

1 + 2 = +3 (start at 1 and move 2 places to the right; end at 3)
2 – 5 = –3 (start at 2 and move 5 places to the left; end at –3)
–2 + 4 = 2 (start at –2 and move 4 places to the right; end at +2)
–1 – 3 = –4 (start at –1 and move 3 places to the left; end at –4)

Remember

The rules are:
+ + makes +
+ – makes –
– + makes –
– – makes +

When two signs appear together, you can replace them by one sign, using mathematical rules. You will learn more about these rules in Module 2.

Change the signs using the rules before you work with the number line.

Examples

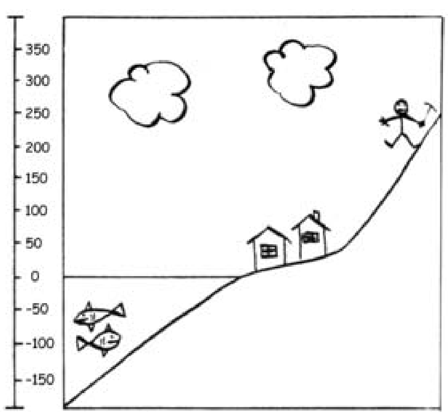
(–3) + (–2) = –3 – 2 = –5 (+ – makes –)
(+3) – (–2) = 3 + 2 = 5 (– – makes +)

Above and below sea level

Directed numbers are also used to give heights above and below sea level. Heights above sea level are positive. Heights below sea level are negative.

Example

What is the difference in height between a point 230 m above sea level and a point 170 m below sea level?



A rough sketch can be helpful when you are trying to solve problems like this one.

Remember

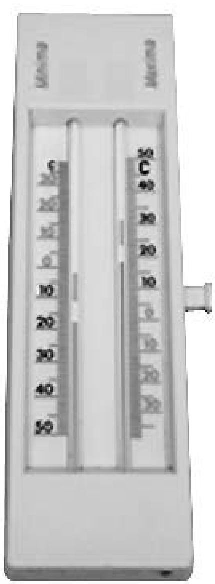
To find the difference, you need to subtract, even if the second number is negative.

230 m above sea level means +230 m.
170 m below sea level means –170 m.

Difference in height = 230 – (–170)
= 230 + 170
= 400 m (remember to include the units of measurement in your answer!)

Exercise

1. Fill in the correct signs (< or >) in place of the * in each example to show which of these heights above and below sea level is the greater.
   a) 150 m * 75 m b) 50 m * –75 m
   c) –65 m * 30 m d) 0 m * –20 m
   e) –10 m * –20 m f) –25 m * 25 m
2. Find the value of:
   a) (+5) + (+3) b) (–4) + (+7)
   c) (+6) + (–2) d) (–1) + (–2)
   e) (+3) – (+8) f) (+2) – (–3)
   g) (–5) – (+1) h) (–5) – (–6)
   i) (–4) + (–3) – (+7).
3. On a certain day at a certain time in Siberia, the temperature was –33 °C. On the same day at the same time, the temperature in Brazil was 33 °C. What is the difference between these two temperatures?



4. The photograph on the left shows you a minimum and maximum thermometer marked in °C. The liquid in the tube pushes an indicator, which remains in place to show the minimum and maximum temperature for a given period. The knob at the side is used to reset the indicators.
   a) What was the temperature at the time this photograph was taken?
   b) What was the minimum temperature experienced on that day?
   c) What was the maximum temperature experienced on that day?
   d) What is the difference between the minimum and maximum temperatures?
5. A submarine is 50 m below sea level.
   a) It goes down a further 280 m to point P. Write down the depth of point P below sea level.
   b) From point P it rises 110 m to point Q. What is the depth of Q below sea level?
6. The temperature on a freezer thermometer shows that food is being stored at –20 °C.
   a) What would the temperature be if it was raised by 5 °C?
   b) What would the temperature be if it was lowered by 0.5 °C every hour for 12 hours?

Fractions

You learnt earlier that fractions are rational numbers that represent parts of a whole. Common or vulgar fractions are written with a numerator and denominator. The line dividing the two parts of the fraction means divide. The same fraction can be expressed in many different ways. For example:

$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10}$ and so on.

These are called equivalent fractions because they are all equal in value. To change a fraction to an equivalent fraction, you multiply the numerator and denominator by the same number.

$\frac{1}{2} = \frac{1}{2} \times \frac{2}{2} = \frac{2}{4}$ (multiply the numerator and denominator by 2)

$\frac{1}{2} = \frac{1}{2} \times \frac{4}{4} = \frac{4}{8}$ (multiply the numerator and denominator by 4)

Remember

Any number divided by itself is 1: $\frac{4}{4} = 1$; $\frac{100}{100} = 1$. So, in fact, to make equivalent fractions, you are simply multiplying by 1.

Note

Changing a fraction to simplest form by dividing is sometimes called cancelling. For example: $\frac{\not{5}}{\not{15}} = \frac{1}{3}$.

You can get an equivalent fraction back to its simplest form by dividing the numerator and denominator by the same number.

In mathematics, fractions are normally written in their simplest form. When the numerator and denominator of a fraction have no common factors, the fraction is said to be in simplest form.

The fractions $\frac{1}{6}$, $\frac{2}{3}$ and $\frac{5}{7}$ are all in simplest form.

Examples

1. Express in simplest form: $\frac{90}{120}$

   $\frac{90}{120} = \frac{90 \div 3}{120 \div 3} = \frac{30}{40} = \frac{3}{4}$

   or $\frac{9\not{0}}{12\not{0}} = \frac{9 \div 3}{12 \div 3} = \frac{3}{4}$

2. Fill in the missing numbers:

   $\frac{1}{2} = \frac{\square}{16}$

   Note: $2 \times 8 = 16$

   $\therefore \frac{1}{2} \times \frac{8}{8} = \frac{8}{16}$

   $\frac{4}{\square} = \frac{12}{15}$

   Note: $12 \div 3 = 4$

   $\therefore \frac{4}{5} = \frac{12 \div 3}{15 \div 3}$

Exercise

1. Write three different equivalent fractions for each of the following given fractions.

   a) $\frac{1}{3}$ b) $\frac{2}{5}$ c) $\frac{5}{7}$

2. Express each of the following fractions in its simplest form.

   a) $\frac{5}{20}$ b) $\frac{16}{24}$ c) $\frac{56}{280}$

3. Find the missing numbers.

   a) $\frac{3}{5} = \frac{\square}{15}$ b) $\frac{3}{\square} = \frac{24}{56}$ c) $\frac{1}{\square} = \frac{25}{100}$

Operations on fractions

Adding or subtracting fractions

When adding or subtracting fractions, the denominators have to be the same. To make denominators the same, you find the LCM of the denominators. Then you add or subtract the numerators and write the answer over the common denominator.

Examples

1. $\frac{3}{7} + \frac{2}{7} = \frac{5}{7}$
2. $\frac{2}{9} + \frac{4}{9} + \frac{1}{9} = \frac{7}{9}$
3. $\frac{1}{2} + \frac{2}{5} = \frac{5}{10} + \frac{4}{10} = \frac{9}{10}$
4. $\frac{7}{12} - \frac{5}{12} = \frac{2}{12} = \frac{1}{6}$
5. $\frac{5}{8} - \frac{1}{20} = \frac{25}{40} - \frac{2}{40} = \frac{23}{40}$
6. $2\frac{1}{4} - 1\frac{7}{10} = \frac{9}{4} - \frac{17}{10} = \frac{45-34}{20} = \frac{11}{20}$

Multiplying fractions

When multiplying fractions, first multiply the numerators. Then multiply the denominators. Write the product of the numerators above the product of the denominators.

Examples

1. $\frac{1}{4} \times 3$

   $= \frac{1}{4} \times \frac{3}{1}$

   $= \frac{3}{4}$

2. $\frac{2}{3} \times \frac{9}{16}$

   $= \frac{{}^{1}\cancel{2} \times \cancel{9}^{3}}{{}^{1}\cancel{3} \times \cancel{16}^{8}}$ (cancel)

   $= \frac{3}{8}$

3. $\frac{2}{3}$ of $1\frac{4}{5}$

   $= \frac{2}{{}_{1}\cancel{3}} \times \frac{\cancel{9}^{3}}{5}$

   $= \frac{6}{5}$

   $= 1\frac{1}{5}$

Exercise

Evaluate the following.

1. $\frac{4}{7} + \frac{2}{7}$
2. $\frac{7}{9} + \frac{5}{9}$
3. $\frac{3}{8} + \frac{1}{4}$
4. $2\frac{1}{7} + 1\frac{3}{6}$
5. $1\frac{4}{9} + 3\frac{5}{12}$
6. $3\frac{2}{9} + 2\frac{1}{3} + 2\frac{7}{12}$
7. $\frac{6}{15} - \frac{4}{15}$
8. $\frac{5}{8} - \frac{1}{4}$
9. $\frac{1}{3} - \frac{1}{5}$
10. $2\frac{5}{8} - 1\frac{1}{4}$
11. $3\frac{1}{4} - 1\frac{3}{8}$
12. $5 - \frac{3}{7}$
13. $5 \times \frac{6}{7}$
14. $\frac{1}{5} \times \frac{2}{3}$
15. $\frac{2}{3} \times \frac{6}{7}$
16. $4\frac{2}{3} \times 1\frac{1}{2}$
17. $\frac{2}{3}$ of 81
18. $\frac{1}{4}$ of $\frac{1}{2}$
19. $2\frac{1}{2}$ of 80
20. $3\frac{1}{7} \times 4\frac{3}{8}$
21. $\frac{2}{3} + \frac{4}{8} \times \frac{1}{2}$
22. $3\frac{1}{2} - 4\frac{1}{3}$
23. $\frac{12}{100} \times \frac{1}{8}$
24. $12 + 3\frac{1}{4}$

> **Hint**
>
> The quick method of finding a reciprocal is to turn the fraction upside down. So the reciprocal of $\frac{3}{4}$ is $\frac{4}{3}$ and the reciprocal of $\frac{1}{4}$ is $\frac{4}{1}$. To find the reciprocal of a whole number, write it as a fraction first. $5 = \frac{5}{1}$ and so the reciprocal of 5 is $\frac{1}{5}$.

Dividing fractions

In order to divide fractions, you need to understand the concept of the reciprocal. The reciprocal or multiplicative inverse of 3 is $\frac{1}{3}$ because $3 \times \frac{1}{3} = \frac{3}{1} \times \frac{1}{3} = 1$. The product of reciprocals is always 1.

To divide a fraction by another fraction, you multiply the first fraction by the reciprocal of the second fraction.

Examples

$\frac{1}{2} \div \frac{1}{4}$
$= \frac{1}{2} \times \frac{4}{1}$
$= \frac{4}{2}$
$= 2$

$\frac{5}{6} \div 6$
$= \frac{5}{6} \times \frac{1}{6}$
$= \frac{5}{36}$

$2\frac{1}{2} \div 3$
$= \frac{5}{2} \div \frac{3}{1}$
$= \frac{5}{2} \times \frac{1}{3}$
$= \frac{5}{6}$

$4\frac{1}{2} \div 6\frac{1}{4}$
$= \frac{9}{2} \div \frac{25}{4}$
$= \frac{9}{{}_{1}\not{2}} \times \frac{{}^{2}\not{4}}{25}$
$= \frac{18}{25}$

Exercise

Evaluate the following.

1. $\frac{2}{3} \div 3$
2. $\frac{3}{4} \div 3$
3. $2\frac{4}{5} \div 7$
4. $7 \div 1\frac{3}{4}$
5. $\frac{7}{12} \div \frac{14}{15}$
6. $4\frac{1}{8} \div 2\frac{3}{4}$
7. $\frac{18}{28} \div \frac{3}{4}$
8. $2\frac{1}{7} \div \frac{2}{14}$
9. $9\frac{1}{3} \div \frac{4}{7}$

Understanding fractions

A newspaper reports that $\frac{1}{3}$ of 45 million people are illiterate in a country. What does this mean? If you remember that 'of' means 'multiply', this is easy!

$\frac{1}{3} \times \frac{45\,000\,000}{1} = 15\,000\,000.$

The newspaper also reports that 40 out of every 50 people interviewed supported the national football team. What fraction is this?

$\frac{4\not{0}}{5\not{0}} = \frac{4}{5}$

The same newspaper reports that 200 people, $\frac{1}{3}$ of all the visitors to a local zoo, complained about the small cages. How many visitors were there altogether?

$\frac{1}{3}$ is 200

This means that $\frac{3}{3}$ (the total) of the number is 3×200.

So the total number is 600.